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Nonlinear parametric oscillations of viscoelastic plate of variable thickness

Нелинейные параметрические колебания вязкоупругой пластинки переменной толщины

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Ключевые слова: тонкостенные конструкции; пластины; оболочки; переменная толщина; периодическая нагрузка; область динамической неустойчивости

Abstract. Isotropic viscoelastic plates of variable thickness under the effect of a uniformly distributed vibration load applied along one of the parallel sides, resulting in parametric resonance (with certain combinations of eigenfrequencies of vibration and excitation forces) are considered in the paper. It is believed that under the effect of this load, the plates undergo the displacements (in particular, deflections) commensurate with their thickness. Geometrically nonlinear mathematical model of the problem of parametric oscillations of a viscoelastic isotropic plate of variable thickness is developed using the classical Kirchhoff-Love hypothesis. Corresponding nonlinear equations of vibration motion of plates under consideration are derived (in displacements). The technique of the nonlinear problem solution by applying the Bubnov-Galerkin method at polynomial approximation of displacements (and deflection) and a numerical method that uses quadrature formula are proposed. The Koltunov-Rzhanitsyn kernel with three different rheological parameters is chosen as a weakly singular kernel. Parametric oscillations of viscoelastic plates of variable thickness under the effect of an external load are investigated. The effect on the domain of dynamic instability of geometric nonlinearity, viscoelastic properties of material, as well as other physical-mechanical and geometric parameters and factors (initial imperfections of the shape, aspect ratios, thickness, boundary conditions, excitation coefficient, rheological parameters) are taken into account. The results obtained are in good agreement with the results and data of other authors. The convergence of the Bubnov-Galerkin method is verified.

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Аннотация. Рассматриваются изотропные вязкоупругие пластинки переменной толщины, находящиеся под действием равномерно распределённой вибрационной нагрузки, приложенной по одной из параллельных сторон, приводящей (при определённых сочетаниях частот собственных колебаний и возмущающей силы) к параметрическому резонансу. Считается, что под воздействием указанной нагрузки пластинки допускают перемещения (в частности, прогибы), соизмеримые с их толщиной. Разработана геометрически нелинейная математическая модель задачи о параметрических колебаниях вязкоупругой изотропной пластинки переменной толщины с использованием классической гипотезы Кирхгофа-Лява. Выведены соответствующие нелинейные уравнения колебательного движения рассматриваемых пластинок (в перемещениях). Предложена методика решения рассматриваемой нелинейной задачи на основе применения метода Бубнова-Галеркина при многочленной аппроксимации перемещений (и прогиба), а также численного метода, использующего квадратурные формулы. В качестве слабо-сингулярного ядра выбрано ядро Колтунова-Ржаницына с тремя различными реологическими параметрами. Исследованы параметрические колебания вязкоупругих пластин переменной толщины под воздействием внешней нагрузки. При этом осуществлялся учёт влияния на области динамической неустойчивости геометрической нелинейности, вязкоупругих свойств материала, а также других физико-механических и геометрических параметров и факторов (начальных несовершенств формы, соотношений сторон, толщины, граничных условий, коэффициента возбуждения, реологических параметров). Полученные результаты хорошо согласуются с результатами и данными других авторов. Сходимость метода Бубнова-Галеркина проверена.

1. Introduction

Construction of the dynamic instability domain of thin-walled elastic systems is a very urgent and important task; a multitude of publications are devoted to this issue, of which, first of all, the studies carried out in [1–6] should be mentioned. Beginning with the article by N.M. Belyaev [1], parametric oscillations become the subject of numerous studies in application to various mechanical systems with distributed parameters, in particular, to rods, plates and shells. In the fundamental monograph V.V. Bolotin [2] has analyzed in detail the occurrence of parametric resonance in elastic thin-walled systems. An extensive monograph by T. Schmidt [6] contains a detailed review of many aspects concerning these issues.

Solution of linear problems of parametric oscillations of thin-walled structures (such as plates and shells) often is reduced to the use of ordinary differential equations with periodic coefficients - to the Mathieu-Hill equations. At certain ratios of coefficients, the solution of this type of equation increases indefinitely, since the corresponding oscillations of thin-walled structures are resonant in nature. The range of parameters that cause these oscillations is called the domain of dynamic instability (DDI). An assessment of the DDI is one of the central problems of parametric oscillations of thin-walled structures.

Solution of the problems connected with the study of resonant phenomena in geometrically nonlinear cases of viscoelastic thin-walled structures of variable thickness has been developed inadequately. The development of a methodology for solving such an important class of problems is of considerable theoretical and practical interest.

Note the following important publications in recent years devoted to the solution of this scientific problem:

1) the A.E. Kalinnikov and N.P. Kuznetsov article [7] presents the algorithm and results of a numerical solution of the problem of transverse oscillations of a straight rod under longitudinal force, harmonically varying in time;

2) the article by Q. Yan, H. Ding, L. Chen [8] contains an analysis of nonlinear parametric oscillations of the Timoshenko viscoelastic beam;

3) the article by L.D. Akulenko, S.A. Kumakshev, S.V. Nesterov [9] is devoted to the study of eigenfrequencies and modes of parametric oscillations of mechanical systems (on the example of small parametric oscillations of a mathematical pendulum of variable length);

4) the article by V.V. Karpov, O.V. Ignat'ev, A.A. Semenov [10] presents an analysis of the stress-strain state of shallow shell structures of double curvature, reinforced from the concave side by a various number of stiffeners. Mindlin-Reissner shell deformation theory is used, which accounts for geometrical nonlinearity and transverse shears, as well as for discrete introduction of stiffeners with contact between the stiffener and the shell along thestrip;

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5) in the article by L. Kurpa, O.S. Mazur, Ya.V. Tkachenko [11] a numerical-analytical method for studying parametric oscillations of plates under the effect of static and dynamic loads (bearing a periodic nature) is substantiated;

6) the articles [12] (M. Darabi, R. Ganesan), [13] (H.Q. Huynh, H. Nguyen, H.L.T. Nguyen), [14] (R. Kumar, S.C. Dutta, S.K. Panda) and [15] (R. Kumar, S. Mondal, Sh. Guchhait, R. Jamatia) contain an analysis of results of studies of dynamic stability of plates (of various types) subjected to harmonic loads (with and without geometrical nonlinearity);

7) in the article by I.D. Ievzerov [16] the stability problems for bars and plates are considered. Variation formulations are used for the stability problem;

8) the article by V.M. Dubrovin and T.A. Butina [17]; the authors propose a method for calculating dynamic stability of a cylindrical shell under axial compressive dynamic load;

9) in the article by A.A. Mochalin [18] on the basis of the V.Z. Vlasov semi-momentless theory, the solution of the problem of dynamic stability of an isotropic cylindrical shell of variable thickness is considered in the direction of its generatrix under external pressure;

10) in the article [19] (M.K. Usarov) is dedicated to improvement of plate theory in order to take into account forces, moments and bimoments, generated by nonlinear law of displacement distribution in plate cross-sections;

11) the article [20] (T. Dey, L.S. Ramachandra) is devoted to parametric oscillations of shells (of various types) subjected to static and dynamic loads of periodic nature (equations of oscillatory motion are obtained by the authors using the Donnell's theory of shell);

12) in the article [21] the author V.M. Budanov has proposed a special linear transformation for expressing the general solution of a second-order differential equation with periodic coefficients; the author has shown with numerical experiments that there exist the periodic solutions of the auxiliary system outside the instability domains of solutions of the Mathieu equation, and the solutions obtained for the DDI are in good agreement with known results;

13) in the article [22] (B.Kh. Eshmatov) is described the analyses of the nonlinear vibrations and dynamic stability of viscoelastic orthotropic plates. The models are based on the Kirchhoff–Love hypothesis and Reissner–Mindlin generalized theory (with the incorporation of shear deformation and rotatory inertia) in geometrically nonlinear statements. With a combination of the Bubnov–Galerkin and the presented method, problems of nonlinear vibrations and dynamic stability in viscoelastic orthotropic rectangular plates have been solved;

14) in the articles [23] (R.A. Abdikarimov, D.P. Goloskokov) and [24] (R.A. Abdikarimov, B.A. Khudayarov) a problem of nonlinear fluctuations of a viscoelastic plate of a variable thickness is considered. The resolving system of the integrodifferential equations of a problem is received. The numerical decision is constructed by Bubnov-Galerkins method with the subsequent application of quadrature formulas for approximation of integrals;

15) in the articles [25, 26] (B.Kh. Eshmatov), accordingly, the dynamic stability and nonlinear vibrations problems of viscoelastic orthotropic and isotropic plates is considered in a geometrically nonlinear formulation using the generalized Timoshenko theory. The problem is solved by the Bubnov-Galerkin procedure combined with a numerical method based on quadrature formulas. The effect of viscoelastic and inhomogeneous properties of the material on the dynamic stability and vibrations of a plate is discussed;

16) in the article [27] (B.Kh. Eshmatov, Kh. Eshmatov, D.A. Khodzhaev) the problem of flutter of viscoelastic rectangular plates and cylindrical panels with concentrated masses is studied in a geometrically nonlinear formulation. The behavior of viscoelastic rectangular plates and cylindrical panels is studied and the critical flow velocities are determined for real composite materials over wide ranges of physicomechanical and geometrical parameters.

Practical value of studies on the assessment of stability of thin-walled structures consists in consideration of the magnitude of existing loads and the law of their variation over time, as well as of all important features of both the structure itself and the properties of materials for their fabrication; this allows a correct estimation of load-bearing capacity of such structures.

Despite the great number of publications devoted to the study of parametric oscillations of thin-walled structures, the number of papers devoted to the study of dynamic stability of these structures with account of viscoelastic properties of their material is few.

2. Methods

Consider a viscoelastic rectangular plate (with sides a and b) of variable thickness $h = h(x, y)$, made of homogeneous isotropic material (Figure 1). Assume that the plate is subjected to an external dynamic load acting along the edge a, having a periodic character: $P(t) = P_0 + P_1 \cos\Theta t$ (here P_0, $P_1 = const$; Θ – the excitation frequency). Also assume that the plate has initial deflections.

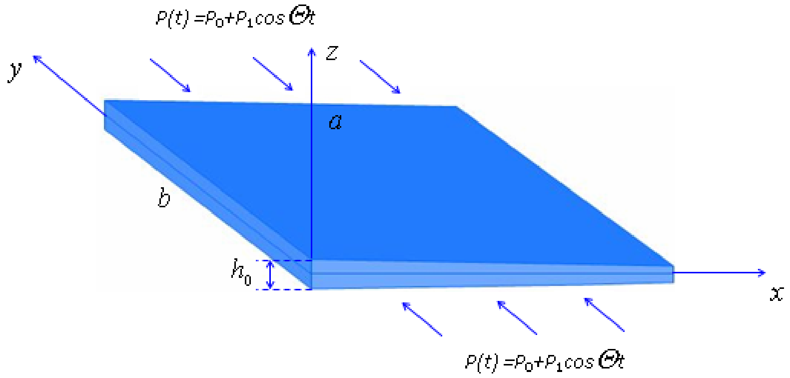

Figure 1. Viscoelastic rectangular plate of variable thickness under action of external periodic load

In [28] R.A. Abdikarimov and V.M. Zhgutov have shown that in this case nonlinear equations of the oscillatory motion of the plate (with respect to deflection $w = w(x, y, t)$ and longitudinal displacements $u = u(x, y, t)$, $v = v(x, y, t)$) under the influence of force $P(t)\frac{\partial^2 w}{\partial x^2}$ with account of initial deflection (at appropriate boundary and initial conditions) can be written in the form:

$$
\begin{gathered}
(1-\Gamma^*)\left[h\left(\frac{\partial \varepsilon_x}{\partial x}+\mu\frac{\partial \varepsilon_y}{\partial x}+\frac{1-\mu}{2}\frac{\partial \gamma_{xy}}{\partial y}\right)+\frac{\partial h}{\partial x}(\varepsilon_x+\mu\varepsilon_y)+\frac{1-\mu}{2}\frac{\partial h}{\partial y}\gamma_{xy}\right]+\\
+\frac{1-\mu^2}{E}p_x-\rho h\frac{1-\mu^2}{E}\frac{\partial^2 u}{\partial t^2}=0\\
(1-\Gamma^*)\left[h\left(\frac{\partial \varepsilon_y}{\partial y}+\mu\frac{\partial \varepsilon_x}{\partial y}+\frac{1-\mu}{2}\frac{\partial \gamma_{xy}}{\partial x}\right)+\frac{\partial h}{\partial y}(\varepsilon_y+\mu\varepsilon_x)+\frac{1-\mu}{2}\frac{\partial h}{\partial x}\gamma_{xy}\right]+\\
+\frac{1-\mu^2}{E}p_y-\rho h\frac{1-\mu^2}{E}\frac{\partial^2 v}{\partial t^2}=0\\
(1-\Gamma^*)\left\{\left[D\nabla^4 w+2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w+2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w+\nabla^2 D\nabla^2 w-\right.\right.\\
\left.\left.-(1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)\right]\right\}+\\
+h\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\sigma_x+\frac{\partial w}{\partial y}\tau_{xy}\right)+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\sigma_y+\frac{\partial w}{\partial x}\tau_{xy}\right)\right]+\frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial x}\sigma_x+\frac{\partial w}{\partial y}\tau_{xy}\right)+\\
+\frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial y}\sigma_y+\frac{\partial w}{\partial x}\tau_{xy}\right)+P(t)\frac{\partial^2 w}{\partial x^2}+q-\rho h\frac{\partial^2 w}{\partial t^2}=0
\end{gathered}
\tag{1}
$$

Introduce to equations (1) the following dimensionless parameters:

$$\bar{u}=\frac{u}{h_0};\ \bar{v}=\frac{v}{h_0};\ \bar{w}=\frac{w}{h_0};\ \bar{w}_0=\frac{w_0}{h_0};\ \bar{x}=\frac{x}{a};\ \bar{y}=\frac{y}{b};\ \bar{h}=\frac{h}{h_0};\ \lambda=\frac{a}{b};\ \delta=\frac{b}{h_0};$$

$$\bar{q}=\frac{q}{E}\left(\frac{b}{h_0}\right)^4;\ \bar{p}_x=\frac{p_x}{E};\ \bar{p}_y=\frac{p_y}{E};\ \frac{\Theta}{\omega};\ \omega t;\ \frac{\Gamma(t)}{\omega};\ \delta_0=\frac{P_0}{P_{кр}};\ \delta_1=\frac{P_1}{P_{кр}}.$$

Using previous notations, a system of integral-differential equations (IDE) in a dimensionless form is obtained.

To solve the system of nonlinear IDE (1), the sought for functions $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$, $w_0(x,y)$ are presented as an expansion in known (basic) functions $\phi_{nm}(x,y)$, $\varphi_{nm}(x,y)$, $\psi_{nm}(x,y)$, $\psi_{0nm}(x,y)$ satisfying the given boundary conditions:

$$\begin{gathered} u(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nm}(t)\phi_{nm}(x,y),\ v(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}v_{nm}(t)\varphi_{nm}(x,y), \\ w(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}w_{nm}(t)\psi_{nm}(x,y),\ w_0(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M}w_{0nm}\psi_{0nm}(x,y) \end{gathered} \qquad (2)$$

where $u_{nm}=u_{nm}(t), v_{nm}=v_{nm}(t)$, $w_{nm}=w_{nm}(t)$ – are unknown time functions; $\varphi_{nm}(x,y)$, $\phi_{nm}(x,y)$, $\psi_{nm}(x,y)$, $\psi_{0nm}(x,y)$, $n=1,2,...,N;\ m=1,2,...,M$ – known (basic) coordinate functions that satisfy the given boundary conditions of the problem (described, for example, in Volmir's monograph [4]).

As a rule, algorithms for solving dynamic problems for viscoelastic thin-walled structures are standard ones. By choosing the basic functions that satisfy certain boundary conditions, the original boundary value problem is reduced to the problem of oscillations of a system with a finite number of degrees of freedom, i.e. to a system of linear (or nonlinear) IDE with one independent variable (of time). As a rule, trigonometric functions or beam functions are used as basic ones. It is known that the choice of such basic functions in many cases limits the class of solved problems to structures of the simplest configurations - beams of constant sections, rectangular plates, cylindrical shells [4].

For elastic structures of a more complex geometric form, it is very tempting to use as basic functions $\varphi_{nm}(x,y)$, $\phi_{nm}(x,y)$, $\psi_{nm}(x,y)$, $\psi_{0nm}(x,y)$ the eigenmodes of oscillations, which are, in a sense, "native" and allow better consideration of all the features of these structures. For the first time, the expansion of the solution in terms of eigenmodes of oscillations in solving problems of described class was used in [29] (by V.N. Chelomei). Subsequently, the eigenmodes of oscillations in the process of expansion of the sought for solution in the problem of forced oscillations of real elastic and viscoelastic structures (including the shell type) were used in [30] (A.N. Ishmatov, M.M. Mirsaidov) and [31] (T.Z. Sultanov, D.A. Khodzhaev, M.M. Mirsaidov).

In [32] (M.M. Mirsaidov, Ya. Mekhmonov), along with expansion in terms of eigenmodes of oscillations, the problem of vibrations of three-dimensional bodies with associated mass was solved. As is well known, the determination of eigenmodes of oscillations for any structures is a completely independent and a very difficult task. That is why quite often a part of a complete system of known functions in analytical form, that satisfies a particular boundary value problem, is used as a basic function.

Substituting (2) into the system (1) written in dimensionless parameters, and following the Bubnov-Galerkin procedure for the functions $u_{nm}(t)$, $v_{nm}(t)$, $w_{nm}(t)$ to be determined, a system of nonlinear IDE, i.e. the basic resolving nonlinear IDE are obtained in the form:

$$\sum_{n=1}^{N}\sum_{m=1}^{M} a_{klnm}\ddot{u}_{nm} - \eta_1\left(1-\Gamma^*\right)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left(d_{1klnm}u_{nm} + \lambda e_{1klnm}v_{nm}\right) + \right.$$

$$\left. + \frac{1}{\delta}\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} g_{1klnmij}\left(w_{nm}w_{ij} - w_{0nm}w_{0ij}\right)\right\} = 0,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}\ddot{v}_{nm} - \eta_2\left(1-\Gamma^*\right)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{\lambda}d_{2klnm}u_{nm} + e_{2klnm}v_{nm}\right) + \right.$$

$$\left. + \frac{1}{\delta}\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} g_{2klnmij}\left(w_{nm}w_{ij} - w_{0nm}w_{0ij}\right)\right\} = 0, \tag{3}$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}\ddot{w}_{nm} + \eta_3\sum_{n=1}^{N}\sum_{m=1}^{M} p_{klnm}^2\left(1-2\mu_{klnm}\cos\Theta t\right)w_{nm} - \eta_3\Gamma^*\sum_{n=1}^{N}\sum_{m=1}^{M} f_{3klnm}w_{nm} -$$

$$-\eta_3\left\{\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} w_{nm}\left(1-\Gamma^*\right)\left(d_{4klnmij}u_{ij} + e_{4klnmij}v_{ij}\right) + \right.$$

$$\left. + \sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M} g_{klnmijrs}w_{nm}\left(1-\Gamma^*\right)w_{ij}w_{rs}\right\} = 12\eta_3\left(1-\mu^2\right)\lambda^4 q_{kl},$$

$$u_{nm}(0) = u_{0nm},\ \dot{u}_{nm}(0) = \dot{u}_{0nm},\ v_{nm}(0) = v_{0nm},\ \dot{v}_{nm}(0) = \dot{v}_{0nm},\ w_{nm}(0) = w_{0nm},$$
$$\dot{w}_{nm}(0) = \dot{w}_{0nm},$$

where $a_{klnm} = \int_0^1\int_0^1 h\phi_{nm}\phi_{kl}\,dxdy$,

$$d_{1klnm} = \int_0^1\int_0^1\left(h\phi''_{nm,xx} + h'_x\phi'_{nm,x} + \lambda^2 h\frac{1-\mu}{2}\phi''_{nm,yy} + \frac{1-\mu}{2}\lambda^2 h'_y\phi'_{nm,y}\right)\phi_{kl}\,dxdy,$$

$$e_{1klnm} = \int_0^1\int_0^1\left(\frac{1+\mu}{2}h\varphi''_{nm,xy} + \mu h'_x\varphi'_{nm,y} + \frac{1-\mu}{2}h'_y\varphi'_{nm,x}\right)\phi_{kl}\,dxdy,$$

$$g_{1klnmij} = \int_0^1\int_0^1\left(\frac{1}{\lambda}h\psi'_{nm,x}\psi'_{ij,xx} + \frac{\lambda(1+\mu)}{2}h\psi'_{nm,y}\psi''_{ij,xy} + \frac{1-\mu}{2}\lambda h\psi'_{nm,x}\psi''_{ij,yy} + \right.$$

$$\left. + \frac{1}{2\lambda}h'_x\psi'_{nm,x}\psi'_{ij,x} + \frac{\mu\lambda}{2}h'_x\psi'_{nm,y}\psi'_{ij,y} + \frac{1-\mu}{2}\lambda h'_y\psi'_{nm,x}\psi'_{ij,y}\right)\phi_{kl}\,dxdy,$$

$$b_{klnm} = \int_0^1\int_0^1 h\varphi_{nm}\varphi_{kl}\,dxdy,$$

$$d_{2klnm} = \int_0^1\int_0^1\left(\frac{1+\mu}{2}h\phi''_{nm,xy} + \frac{1-\mu}{2}h'_x\phi'_{nm,y} + \mu h'_y\phi'_{nm,x}\right)\varphi_{kl}\,dxdy,$$

$$e_{2klnm} = \int_0^1\int_0^1\left(\frac{1-\mu}{2\lambda^2}h\phi''_{nm,xx} + h\varphi''_{nm,yy} + \frac{1-\mu}{2}\frac{1}{\lambda^2}h'_x\varphi'_{nm,x} + h'_y\varphi'_{nm,y}\right)\varphi_{kl}\,dxdy,$$

$$g_{2klnmij} = \int_0^1\int_0^1 \Big(h\psi'_{nm,y}\psi''_{ij,yy} + \frac{1+\mu}{2\lambda^2} h\psi'_{nm,x}\psi''_{ij,xy} + \frac{1-\mu}{2\lambda^2} h\psi'_{nm,y}\psi''_{ij,xx} +$$

$$+ \frac{1-\mu}{2\lambda^2} h'_x\, \psi'_{nm,x}\psi'_{ij,y} + \frac{1}{2} h'_y\, \psi'_{nm,y}\psi'_{ij,y} + \frac{\mu}{2\lambda^2} h'_y\, \psi'_{nm,x}\psi'_{ij,x} \Big)\varphi_{kl}\,dxdy\,,$$

$$c_{klnm} = \int_0^1\int_0^1 h\psi_{nm}\psi_{kl}\,dxdy,\;\; p^2_{klnm} = f_{klnm} - 4\pi^2\lambda^2 p^*_{klnm}\delta_0\,,\;\; \mu_{k\ln m} = \frac{2\pi^2\lambda^2 p^*_{k\ln m}}{p^2_{k\ln m}}\delta_1\,,$$

$$f_{3klnm} = \int_0^1\int_0^1 \Big\{ h^3\Big(\psi^{IV}_{nm,xxxx} + 2\lambda^2\psi^{IV}_{nm,xxyy} + \lambda^4\psi^{IV}_{nm,yyyy}\Big) +$$

$$+ 3\Big[2h\big(h'_x\big)^2 + h^2h''_{xx}\Big]\Big(\psi''_{nm,xx} + \lambda^2\mu\psi''_{nm,yy}\Big) + 6h^2h'_x\,\Big(\psi'''_{nm,xxx} + \lambda^2\psi'''_{nm,xyy}\Big) +$$

$$+ 6h^2h'_y\,\Big(\lambda^4\psi'''_{nm,yyy} + \lambda^2\psi'''_{nm,xxy}\Big) + 3\Big[2h\big(h'_y\big)^2 + h^2h''_{yy}\Big]\Big(\lambda^4\psi''_{nm,yy} + \lambda^2\mu\psi''_{nm,xx}\Big) +$$

$$+ 6(1-\mu)\lambda^2\Big[2hh'_x\,h'_y + h^2h''_{xy}\Big]\psi''_{nm,xy}\Big\}\psi_{kl}\,dxdy\,,$$

$$d_{4klnmij} = 12\int_0^1\int_0^1 \Big\{ \lambda\delta h\psi'_{nm,x}\phi''_{ij,xx} + \frac{1-\mu}{2}\lambda^3\delta h\psi'_{nm,x}\phi''_{ij,yy} + \lambda\delta h'_x\,\psi'_{nm,x}\phi'_{ij,x} +$$

$$+ \frac{1-\mu}{2} h'_y\lambda^3\delta\psi'_{nm,x}\phi'_{ij,y} + \lambda\delta h\,\psi''_{nm,xx}\phi'_{ij,x} + \frac{1+\mu}{2}\lambda^3\delta h\,\psi'_{nm,y}\phi''_{ij,xy} + \lambda^3\delta\mu h'_y\psi'_{nm,y}\phi'_{ij,x} +$$

$$+ \frac{1-\mu}{2}\lambda^3\delta h'_x\,\psi'_{nm,y}\phi'_{ij,y} + \lambda^3\delta\mu h\,\psi''_{nm,yy}\phi'_{ij,x} + (1-\mu)\lambda^3\delta h\,\psi''_{nm,xy}\phi'_{ij,y}\Big\}\psi_{kl}\,dxdy\,,$$

$$e_{4klnmij} = 12\int_0^1\int_0^1 \Big\{ \frac{1+\mu}{2}\lambda^2\delta h\psi'_{nm,x}\varphi''_{ij,xy} + \lambda^2\delta\mu h'_x\,\psi'_{nm,x}\varphi'_{ij,y} + \frac{1-\mu}{2}\lambda^2\delta h'_y\,\psi'_{nm,x}\varphi'_{ij,x} +$$

$$+ \lambda^2\delta\mu h\,\psi''_{nm,xx}\varphi'_{ij,y} + \lambda^4\delta h\,\psi'_{nm,y}\varphi''_{ij,yy} + \frac{1-\mu}{2}\lambda^2\delta h\,\psi'_{nm,y}\varphi''_{ij,xx} + \lambda^4\delta h'_y\psi'_{nm,y}\varphi'_{ij,y} +$$

$$+ \frac{1-\mu}{2}\lambda^2\delta h'_x\,\psi'_{nm,y}\varphi'_{ij,x} + \lambda^4\delta h\,\psi''_{nm,yy}\varphi'_{ij,y} + (1-\mu)\lambda^2\delta h\,\psi''_{nm,xy}\varphi'_{ij,x}\Big\}\psi_{kl}\,dxdy\,,$$

$$g_{klnmijrs} = 12\int_0^1\int_0^1 \Big\{ h\psi'_{nm,x}\psi'_{ij,x}\psi''_{rs,xx} + \frac{1-\mu}{2}\lambda^2 h\psi'_{nm,x}\psi'_{ij,x}\psi''_{rs,yy} +$$

$$+ \frac{1+\mu}{2}\lambda^2 h\psi'_{nm,x}\psi'_{ij,y}\psi''_{rs,xy} + \frac{1}{2}h'_x\psi'_{nm,x}\psi'_{ij,x}\psi'_{rs,x} + \frac{\lambda^2\mu}{2}h'_x\psi'_{nm,x}\psi'_{ij,y}\psi'_{rs,y} +$$

$$+ \frac{1-\mu}{2}h'_y\lambda^2\psi'_{nm,x}\psi'_{ij,x}\psi'_{rs,y} + \frac{1}{2}h\psi''_{nm,xx}\psi'_{ij,x}\psi'_{rs,x} + \frac{\lambda^2\mu}{2}h\psi''_{nm,xx}\psi'_{ij,y}\psi'_{rs,y} +$$

$$+ \lambda^4 h\psi'_{nm,y}\psi''_{ij,yy}\psi'_{rs,y} + \frac{1-\mu}{2}\lambda^2 h\psi'_{nm,y}\psi''_{ij,xx}\psi'_{rs,y} + \frac{1+\mu}{2}\lambda^2 h\psi'_{nm,y}\psi''_{ij,xy}\psi'_{rs,x} +$$

$$+ \frac{\lambda^4}{2}h'_y\psi'_{nm,y}\psi'_{ij,y}\psi'_{rs,y} + \frac{\lambda^2\mu}{2}h'_y\psi'_{nm,y}\psi'_{ij,x}\psi'_{rs,x} + \frac{1-\mu}{2}\lambda^2 h'_x\psi'_{nm,y}\psi'_{ij,x}\psi'_{rs,y} +$$

$$+ \frac{\lambda^4}{2}h\psi''_{nm,yy}\psi'_{ij,y}\psi'_{rs,y} + \frac{\lambda^2\mu}{2}h\psi''_{nm,yy}\psi'_{ij,x}\psi'_{rs,x} +$$

$$+ (1-\mu)\lambda^2 h\psi''_{nm,xy}\psi'_{ij,y}\psi'_{rs,y}\Big\}\psi_{kl}\,dxdy\,,$$

$$q_{kl} = q\int_0^1\int_0^1 \psi_{kl}dxdy\ ,\ \eta_1 = \frac{3\delta^2}{\pi^4\lambda^2};\ \eta_2 = \frac{3\delta^2}{\pi^4};\ \eta_3 = \frac{1}{4\pi^4\lambda^4}.$$

Twice integrating the obtained system (3) with respect to time t, it can be written in integral form, as shown in the articles by A.F. Verlan, R.A. Abdikarimov, H. Eshmatov [33].

Assuming that $t = t_i,\ t_i = i\Delta t,\ i = 1,2,...$ (here Δt is the step of numerical integration) and replacing the integrals with quadrature formulas of trapezoid for calculating the unknowns $u_{inm} = u_{inm}(t_i)$, $v_{inm} = v_{inm}(t_i)$, $w_{inm} = w_{inm}(t_i)$, the following recurrence formulas are obtained for the Koltunov-Rzhanitsyn kernel $\Gamma(t) = Ae^{-\beta t}\cdot t^{\alpha-1}$, $A > 0,\ \beta > 0$, $0 < \alpha < 1$, described in the monograph by M.A. Koltunov, A.S. Kravchuk, V.P. Mayboroda [34] and in A.R. Rzhanitsyn's monograph [35].

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{m=1}^{M} a_{klnm}u_{pnm} = \sum_{n=1}^{N}\sum_{m=1}^{M} a_{klnm}\left(u_{0nm} + \dot{u}_{0nm}t_p\right) + \eta_1\sum_{q=0}^{p-1}A_q\left(t_p - t_q\right)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M} d_{1klnm}\left(u_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}u_{q-z,nm}\right) + \right.\\
&\qquad + \lambda\sum_{n=1}^{N}\sum_{m=1}^{M} e_{1klnm}\left(v_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}v_{q-z,nm}\right) + \\
&\qquad \left. + \frac{1}{\delta}\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} g_{1klnmij}\left(w_{qnm}w_{qij} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}w_{q-z,nm}w_{q-z,ij}\right)\right\},\\
&\sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}v_{pnm} = \sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}\left(v_{0nm} + \dot{v}_{0nm}t_p\right) + \eta_2\sum_{q=0}^{p-1}A_q\left(t_p - t_q\right)\left\{\frac{1}{\lambda}\sum_{n=1}^{N}\sum_{m=1}^{M} d_{2klnm}\left(u_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}u_{q-z,nm}\right) + \right.\\
&\qquad + \eta_2\sum_{q=0}^{p-1}A_q\left(t_p - t_q\right)\left\{\frac{1}{\lambda}\sum_{n=1}^{N}\sum_{m=1}^{M} d_{2klnm}\left(u_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}u_{q-z,nm}\right) + \right.\\
&\qquad + \sum_{n=1}^{N}\sum_{m=1}^{M} e_{2klnm}\left(v_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}v_{q-z,nm}\right) + \\
&\qquad \left. + \frac{1}{\delta}\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} g_{2klnmij}\left(w_{qnm}w_{qij} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}w_{q-z,nm}w_{q-z,ij}\right)\right\}, \qquad (4)\\
&\sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}w_{pnm} = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}\left(w_{0nm} + \dot{w}_{0nm}t_p\right) - \eta_3\sum_{n=1}^{N}\sum_{m=1}^{M} p_{k\ln m}^2\left(1 - 2\mu_{k\ln m}\cos\Theta t\right)w_{nm} - \\
&\qquad - \eta_3\sum_{q=0}^{p-1}A_q\left(t_p - t_q\right)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M} f_{3klnm}\left(w_{qnm} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}w_{q-z,nm}\right) - \right.\\
&\qquad - \sum_{n,i=1}^{N}\sum_{m,j=1}^{M} d_{4klnmij}w_{qnm}\left(u_{qij} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}u_{q-z,ij}\right) - \\
&\qquad - \sum_{n,i=1}^{N}\sum_{m,j=1}^{M} e_{4klnmij}w_{qnm}\left(v_{qij} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}v_{q-z,ij}\right) - \\
&\left. - \sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M} g_{klnmijrs}w_{qnm}\left(w_{qij}w_{qrs} - \frac{A}{\alpha}\sum_{z=0}^{q}B_z e^{-\beta t_z}w_{q-z,ij}w_{q-z,rs}\right) - 12\left(1-\mu^2\right)\lambda^4 q_{kl}\right\},\\
&\qquad u_{nm}(0) = u_{0nm},\ \dot{u}_{nm}(0) = \dot{u}_{0nm},\ v_{nm}(0) = v_{0nm},\ \dot{v}_{nm}(0) = \dot{v}_{0nm},\\
&\qquad\qquad w_{nm}(0) = w_{0nm},\ \dot{w}_{nm}(0) = \dot{w}_{0nm}.
\end{aligned}$$

Here A_q, B_z – are numerical coefficients that do not depend on the choice of integrands and could be of different values depending on the quadrature formulas used.

3. Results and Discussion

Results of computational experiments corresponding to different physical and geometric parameters of a viscoelastic isotropic plate are illustrated by the graphs shown in Figures 3-7. The dependence of the change in thickness is taken in the form

$$h(x)=\frac{1}{2}h_0\left(1-\alpha^* x\right).$$

Here $h_0 = h(0) = const$, $\alpha^* -$ is a parameter characterizing the change in thickness. Note that this law leads to a liniar reduction in thickness of structural element in direction of Ox axis (Figure 2).

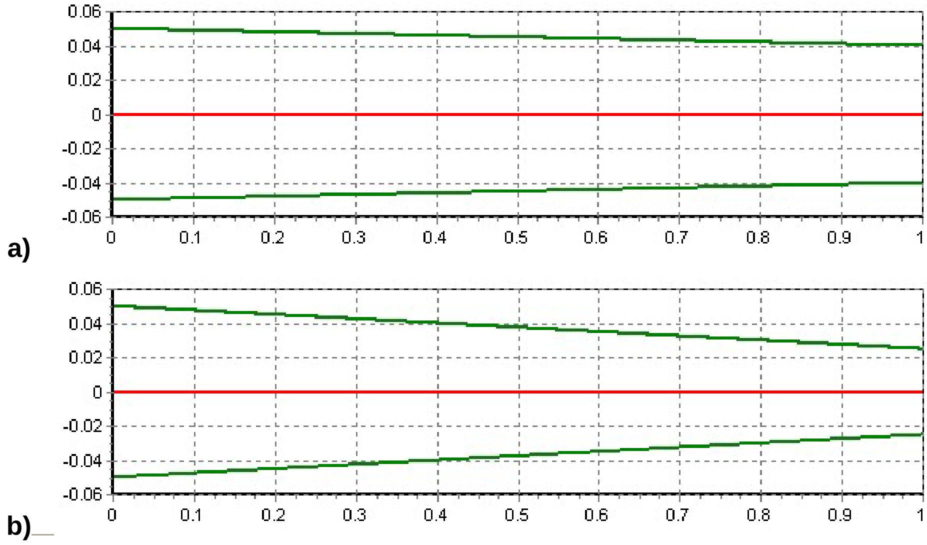

Figure 2. Changes in plate thickness depending on α^* parameter:
a) $\alpha^* = 0.2$; b) $\alpha^* = 0.5$

Figures 3 and 4 illustrate the effect of viscoelastic properties of material on dynamic behavior of the plate.

For example, Figure 3 shows the time t variation of displacements of the midpoint ($x = 0.5$, $y = 0.5$) for different values of rheological parameter A of the relaxation kernel of material. An analysis of results has shown that in all the cases considered, at initial time, the oscillations occur near the equilibrium (initial) state. Over time, the amplitude slowly increases and reaches the value comparable with the one of plate thickness.

Curve 1 ($A = 0$) corresponds to an elastic solution (without considering viscoelastic properties of material); here the oscillations occur without attenuation. Note that in this case the amplitude of oscillations does not increase, but the beating occurs, which is explained by the proximity of the frequency of natural oscillations of the plate to the frequency of periodic effects.

Curve 2 ($A = 0.5$) and curve 3 ($A = 0.1$) are obtained considering the viscoelastic properties of material. It is seen that damped oscillations occur near initial oscillations, but over time, oscillations occur with practically constant amplitude, i.e. become steady. In this case, there is some deviation of oscillations from the neutral axis, which is explained by the presence of initial imperfections in the shape.

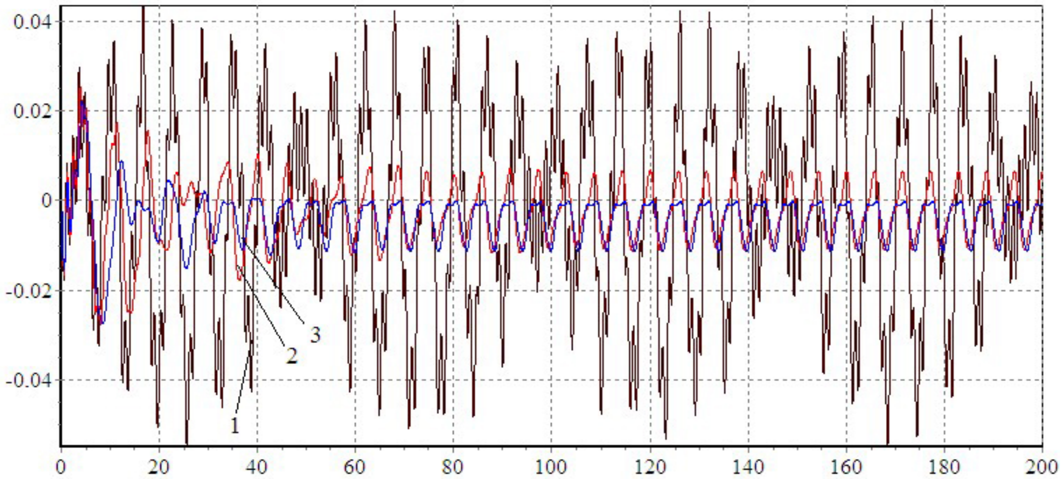

Figure 3. Graphs of variation with time t of displacements of the midpoint ($x = 0.5$, $y = 0.5$) of the plate at different values of the relaxation kernel parameter A:

$\alpha = 0.25$; $\beta = 0.05$; $\mu = 0.3$; $\delta = 25$; $w_0 = 0.01$; $q = 0$; $\lambda = 1$; $\alpha^* = 0.5$; $\delta_0 = 0$; $\delta_1 = 0.5$; $\Theta_1 = 1.1$; $A = 0.0$ **(1);** 0.05 **(2);** 0.1 **(3)**

Figure 4 shows the graphs of deflection versus time t for the midpoint ($x = 0.5$, $y = 0.5$) of the plate corresponding to different values of the relaxation kernel parameter: at $\alpha = 0.1$ (curve 1); at $\alpha = 0.25$ (curve 2); at $\alpha = 0.5$ (curve 3).

In all considered cases, oscillations occur near the equilibrium state in the proximity of initial time. Over time, the amplitude slowly increases, and reaches the thickness of the plate. Then the pattern described is periodically repeated at values of $\alpha = 0.25; 0.5$. At sufficiently large values of α and time, the amplitude of oscillations decreases insignificantly. The amplitude of oscillations at $\alpha = 0.1$ at large values of time increases, which is explained by the proximity to the resonance phenomenon, since the difference between the eigenfrequency of vibrations of the plate and the frequency of excitations becomes smaller, which completely corresponds to results described in the monograph by A.E. Bogdanovich [5].

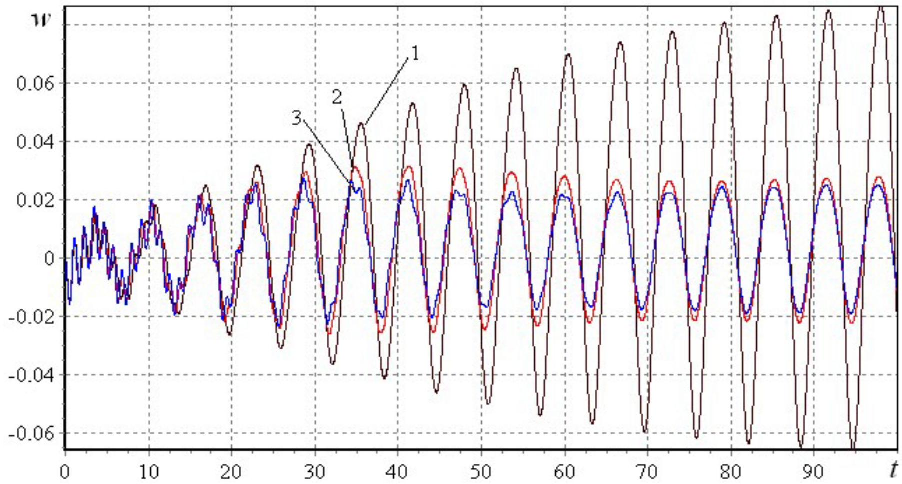

Figure 4. Graphs of dependence of deflection of the midpoint ($x = 0.5$, $y = 0.5$) of the plate on time t at different values of the relaxation kernel parameter α:

$A = 0.01$; $\beta = 0.05$; $\mu = 0.3$; $\delta = 25$; $w_0 = 0.01$; $q = 0$; $\lambda = 1$; $\alpha^* = 0.5$; $\delta_0 = 0$; $\delta_1 = 0.5$; $\Theta_1 = 1.1$; $\alpha = 0.1$ **(1);** 0.25 **(2);** 0.5 **(3)**

Figure 5 shows the graphs of deflection versus time t of the midpoint ($x = 0.5$, $y = 0.5$) of the plate for different aspect ratios: at $\lambda = 1.0$ (curve 1); at 1.5 (curve 2); at 2 (curve 3). Results of calculations show that for a square plate (with aspect ratio $\lambda = 1.0$), and for a rectangular plate with $\lambda = 1.5$ at initial time, oscillations occur near the equilibrium state and, over time, approach harmonic oscillations. At $\lambda = 2.0$, oscillations at initial period of time occur near initial state w_0, and over time, oscillation amplitudes increase sharply, which is explained by the approach of eigenfrequency of the plate to the frequency of excitation force.

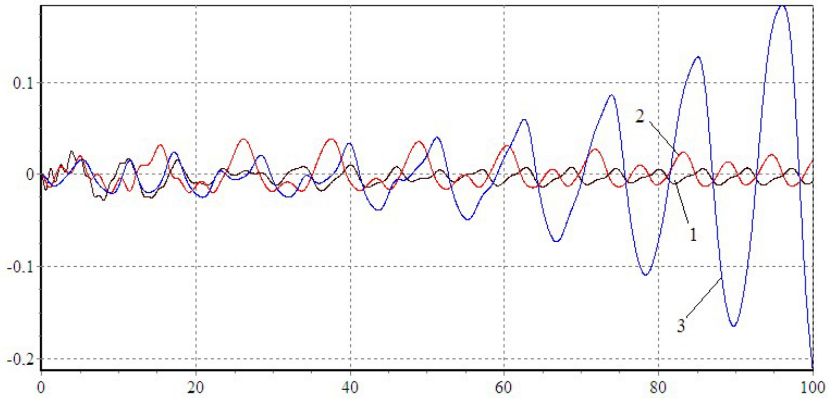

Figure 5. Graphs of dependence of deflection on time t of the midpoint ($x = 0.5$, $y = 0.5$) of the plate at different aspect ratios λ:

$A = 0.05$; $\alpha = 0.25$; $\beta = 0.05$; $\mu = 0.3$; $\delta = 25$; $w_0 = 0.001$; $q = 0$; $\alpha^* = 0.5$; $\delta_0 = 0$; $\delta_1 = 0.5$; $\Theta_1 = 1.1$; $\lambda = 1$ **(1);** 1.5 **(2);** 2 **(3)**

Figure 6 shows the graphs of deflection versus time t for the midpoint ($x = 0.5$, $y = 0.5$) of the plate at various values of its thickness parameter α^*, taking into account viscoelastic properties of material. Namely: at $\alpha^* = 0$ (curve 1); at 0.4 (curve 2); at 0.8 (curve 3). Note that an increase in the parameter of α^* according to the above law corresponds to a decrease in thickness of the plate along the length of Ox axis. This causes the resonance regime and an increase in oscillation amplitude of the plate (curve 3). At other values of parameter α^* (curves 1 and 2), the oscillations become steady.

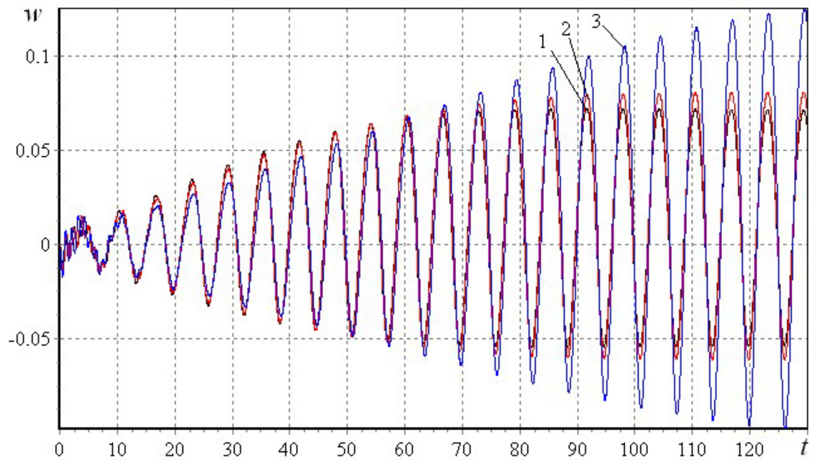

Figure 6. Graphs of dependence of deflection on time t of the midpoint ($x = 0.5$, $y = 0.5$) of the plate at various values of the parameter of its thickness α^*: $A = 0.05$; $\alpha = 0.25$; $\beta = 0.05$; $\mu = 0.3$; $\delta = 25$; $\lambda = 1$; $w_0 = 0.01$; $q = 0$; $\delta_0 = 0$; $\delta_1 = 0.5$; $\Theta_1 = 1.1$; $\alpha^* = 0$ **(1);** 0.4 **(2);** 0.8 **(3)**

Figure 7 presents results of studies on vibrations of a viscoelastic plate at various values of external constant loads: at $q = 0.2$ (curve 1); at $q = 0.5$ (curve 2); at $q = 1$ (curve 3). It is seen that as external load increases, oscillations occur near the equilibrium state of the plate (Figure 7). At initial period of time, the nonlinear parametric oscillations retain the beat character, with the passage of time they become less pronounced, and approach stationary ones. Analogous qualitative results are described in Bogdanovich's monograph [5].

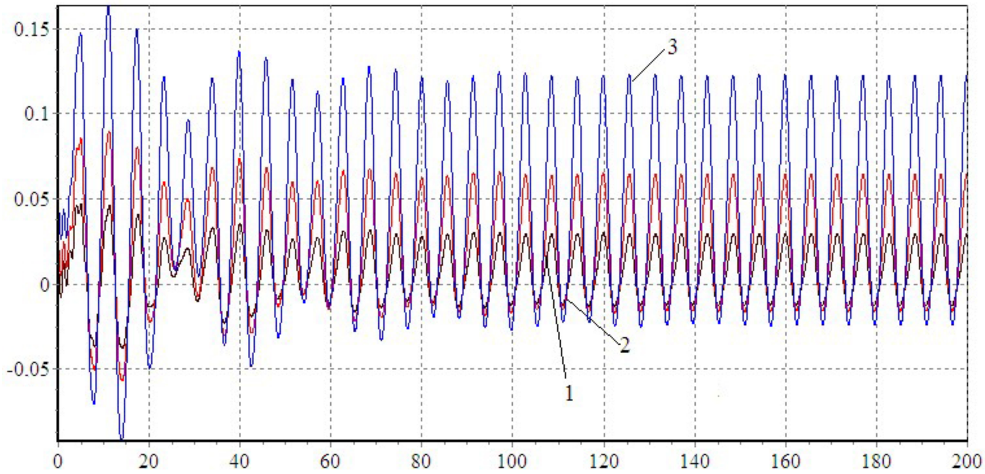

Figure 7. Graphs of dependence of deflection on time t for the midpoint ($x = 0.5$, $y = 0.5$) of the plate at different values of external load q:

$A = 0.05$; $\alpha = 0.25$; $\beta = 0.05$; $\mu = 0.3$; $\delta = 25$; $\lambda = 1$; $w_0 = 0.01$; $\alpha^* = 0.5$; $\delta_0 = 0$; $\delta_1 = 0.5$; $\Theta_1 = 1.1$; $q = 0.2$ **(1);** 0.5 **(2);** 1 **(3)**

4. Conclusions

1. Mathematical model, technique for its investigation and computational algorithm for the problem of parametric nonlinear oscillations of plates of variable thickness are developed taking into account viscoelastic properties of material.

2. An appropriate computer software package is designed to effectively evaluate dynamic instability of plates in question, depending on mechanical characteristics of their material, external excitation loads, geometric parameters, eigenfrequency of vibration, and excitation forces.

3. As a result of numerical experiments it was established that:

a. the presence of viscoelastic properties of material leads to a steady-state oscillation regime even in resonance modes;

b. the presence of initial imperfections of the form leads to a deviation from the reference plane, which is assumed to be neutral (due to the deviation of the latter);

c. a decrease in thickness of the plate along its length in some cases can lead to a resonance mode of oscillations.

Besides, the dependences of some parameters of stable oscillatory process of the plates on the amplitude and frequency of external transverse load are obtained.

References

1. Belyaev, N.M. Ustoychivost prizmaticheskikh sterzhney pod deystviyem peremennykh prodolnykh sil [Stability of Prismatic Rods under the Influence of Variable Longitudinal Forces]. Engineering Constructions and Structural Mechanics. 1924. P. 149-167. (rus)
2. Bolotin, V.V. Dinamicheskaya ustoychivost uprugikh sistem [Dynamic Stability of Elastic Systems]. Moscow: Gostekhizdat. 1956. 600 p. (rus)
3. Vlasov, V.Z. Obshchaya teoriya obolochek i yeye prilozheniya v tekhnike [General Theory of Shells and its Applications in Engineering]. Moscow-Leningrad: Gostekhizdat. 1949. 784 p. (rus)

Литература

1. Беляев Н.М. Устойчивость призматических стержней под действием переменных продольных сил // Инженерные сооружения и строительная механика. 1924. С. 149–167.
2. Болотин В.В. Динамическая устойчивость упругих систем. М.: Гозтехиздат, 1956. 600 с.
3. Власов В.З. Общая теория оболочек и ее приложения в технике. М.-Л.: Гостехиздат. 1949. 784 с.
4. Вольмир А.С. Нелинейная динамика пластинок и оболочек. М.: Наука. 1972. 432 с.
5. Богданович А.Е. Нелинейные задачи динамики цилиндрических композитных оболочек. Рига: Зинатне, 1987. 295 с.

4. Volmir, A.S. Nelineynaya dinamika plastinok i obolochek [Nonlinear Dynamics of Plates and Shells]. Moscow: Science, 1972. 432 p. (rus)
5. Bogdanovich, A.E. Nelineynyye zadachi dinamiki tsilindricheskikh kompozitnykh obolochek [Nonlinear Problems of the Dynamics of Cylindrical Composite Shells]. Riga: Zinatne, 1987. 295 p. (rus)
6. Schmidt, T. Parametricheskiye kolebaniya [Parametric Oscillations]. Moscow: Mir, 1978. 336 p. (rus)
7. Kalinnikov, A.E., Kuznetsov, N.P. Chislennoye issledovaniye parametricheskikh kolebaniy sterzhnya [Numerical Investigation of Parametrical Vibrations of Beam]. Vestnik M.T. Kalashnikov IZHGTU. 2016. Vol. 69. No. 1. Pp. 3–6. (rus)
8. Yan, Q., Ding, H., Chen, L. Nonlinear dynamics of axially moving viscoelastic timoshenko beam under parametric and external excitations. Applied Mathematics and Mechanics. 2015. Vol. 36. No. 8. Pp. 971–984.
9. Akulenko, L.D., Kumakshev, S.A., Nesterov, S.V. Chislenno-analiticheskiy metod issledovaniya parametricheskikh kolebaniy [A Numerical-analytical Method for Investigating Parametric Oscillations]. Journal of Applied Mathematics and Mechanics. 2015. Vol. 79. No. 2. P. 111–121. (rus)
10. Karpov, V.V., Ignat'ev, O.V., Semenov A.A. The stress-strain state of ribbed shell structures. Magazine of Civil Engineering. 2017. No. 6. Pp. 147–160.
11. Kurpa, L., Mazur, O.S., Tkachenko, Ya.V. Parametric Vibration of Multilayer Plates of Complex Shape. Journal of Mathematical Sciences. 2014. Vol. 203. No. 2. Pp. 165–184.
12. Darabi, M., Ganesan, R. Nonlinear dynamic instability analysis of laminated composite thin plates subjected to periodic in-plane loads. Nonlinear Dynamics. 2018. Vol. 91. No. 1. Pp. 187–215.
13. Huynh, H.Q., Nguyen, H., Nguyen, H.L.T. Non–linear parametric vibration and dynamic instability of laminated composite plates using extended dynamic stiffness method. Journal of Engineering Technology. 2017. Vol. 6. Pp. 170–185.
14. Kumar, R., Dutta, S.C., Panda S.K. linear and non–linear dynamic instability of functionally graded plate subjected to non–uniform loading. Composite Structures. 2016. Vol. 154. Pp. 219–230.
15. Kumar, R., Mondal, S., Guchhait, Sh., Jamatia, R. Analytical approach for dynamic instability analysis of functionally graded skew plate under periodic axial compression. International Journal of Mechanical Sciences. 2017. Vol. 130. Pp. 41–51.
16. Ievzerov, I.D. The stability problems for bars and plates. Magazine of Civil Engineering. 2014. No. 1. Pp. 6–11. doi: 10.5862/MCE.45.2 (rus)
17. Dubrovin, V.M., Butina, T.A. Modelirovaniye dinamicheskoy ustoychivosti tsilindricheskoy obolochki pri tsiklicheskom osevom vozdeystvii [Simulation of Dynamic Stability of a Cylindrical Shell under Cyclic Axial Impact]. Mathematical Modeling and Computational Methods. 2016. No. 3(11). Pp. 24–32. (rus)
18. Mochalin, A.A. Parametricheskiye kolebaniya neodnorodnoy krugovoy tsilindricheskoy obolochki peremennoy plotnosti pri razlichnykh krayevykh usloviyakh [The Parametric Oscillations of Heterogeneous Circular Cylindrical Shell of Variable Density under Different Boundary Conditions]. Izv. Saratov Univ. Ser. Math. Mech. Inform. 2015. Vol. 15. No. 2. Pp. 210–215. (rus)
19. Usarov, M.K. [Buckling of orthotropic plates with bimoments. Magazine of Civil Engineering. 2015. No. 1. Pp. 80–90. (rus)
20. Dey, T., Ramachandra, L.S. Dynamic stability of simply supported composite cylindrical shells under partial axial

6. Шмидт Т. Параметрические колебания. М.: Мир, 1978. 336 с.
7. Калинников А.Е., Кузнецов Н.П. Численное исследование параметрических колебаний стержня // Вестник ИЖГТУ им. М.Т.Калашникова. 2016. Т. 69. № 1. С. 3–6.
8. Yan Q., Ding H., Chen L. Nonlinear dynamics of axially moving viscoelastic Timoshenko beam under parametric and external excitations // Applied Mathematics and Mechanics. 2015. Vol. 36. № 8. Pp. 971–984.
9. Акуленко Л.Д., Кумакшев С.А., Нестеров С.В. Численно–аналитический метод исследования параметрических колебаний // Прикладная математика и механика. 2015. Т. 79. № 2. С. 163–180.
10. Карпов В.В., Игнатьев О.В., Семенов А.А. Напряженно-деформированное состояние ребристых оболочечных конструкций // Инженерно-строительный журнал. 2017. № 6(74). С. 147–160.
11. Kurpa L., Mazur O.S. Tkachenko Ya.V. Parametric Vibration of Multilayer Plates of Complex Shape // Journal of Mathematical Sciences. 2014. Vol. 203. № 2. Pp. 165–184.
12. Darabi M., Ganesan R. Nonlinear dynamic instability analysis of laminated composite thin plates subjected to periodic in–plane loads // Nonlinear Dynamics. 2018. Vol. 91. № 1. Pp. 187–215.
13. Huynh H.Q., Nguyen H., Nguyen H.L.T. Non–linear parametric vibration and dynamic instability of laminated composite plates using extended dynamic stiffness method // Journal of Engineering Technology. 2017. Vol. 6. Pp. 170–185.
14. Kumar R., Dutta S.C., Panda S.K. Linear and non–linear dynamic instability of functionally graded plate subjected to non–uniform loading // Composite Structures. 2016. Vol. 154. Pp. 219–230.
15. Kumar R., Mondal S., Guchhait Sh., Jamatia R. Analytical approach for dynamic instability analysis of functionally graded skew plate under periodic axial compression // International Journal of Mechanical Sciences. 2017. Vol. 130. Pp. 41–51.
16. Евзеров И.Д. Задачи устойчивости для стержней и пластин // Инженерно–строительный журнал. 2014. № 1(45). С. 6–11.
17. Дубровин В.М., Бутина Т.А. Моделирование динамической устойчивости цилиндрической оболочки при циклическом осевом воздействии // Математическое моделирование и численные методы. 2016. № 3(11). С. 24–32.
18. Мочалин А.А. Параметрические колебания неоднородной круговой цилиндрической оболочки переменной плотности при различных краевых условиях // Изв. Сарат. ун–та. Нов. сер. Сер. Математика. Механика. Информатика. 2015. Т. 15. № 2. С. 210–215.
19. Усаров М.К. Изгиб ортотропных пластин с учетом бимоментов // Инженерно–строительный журнал. 2015. № 1(53). С. 80–90.
20. Dey T., Ramachandra L.S. Dynamic stability of simply supported composite cylindrical shells under partial axial loading // Journal of Sound and Vibration. 2015. Vol. 353. Pp. 272–291.
21. Budanov V.M. A form of the solution to the Mathieu equation // Moscow University Mechanics Bulletin. 2017. Vol. 72. № 1. Pp. 7–12.
22. Eshmatov B.Kh. Nonlinear vibrations and dynamic stability of viscoelastic orthotropic rectangular plates // Journal Sound and Vibration. 2007. Vol. 300. № 3. Pp. 709–726.
23. Абдикаримов Р.А., Голоскоков Д.П. Численное исследование нелинейных колебаний вязкоупругих

loading. Journal of Sound and Vibration. 2015. Vol. 353. Pp. 272–291.

21. Budanov, V.M. A form of the solution to the Mathieu equation. Moscow University, Mechanics Bulletin. 2017. Vol. 72. No. 1. Pp. 7–12.
22. Eshmatov, B.Kh. Nonlinear vibrations and dynamic stability of viscoelastic orthotropic rectangular plates. Journal Sound and Vibration. 2007. Vol. 300. No. 3. Pp. 709–726.
23. Abdikarimov, R.A., Goloskokov, D.P. Chislennoye issledovaniye nelineynykh kolebaniy vyazkouprugikh plastin peremennoy tolshchiny [Numerical research of nonlinear fluctuations of viscoelastic plates of the variable thickness]. The Journal of University of Water Communications. 2011. No. 2(9). Pp. 102–107. (rus)
24. Abdikarimov, R.A., Khudayarov, B.A. Modelirovaniye kolebatelnykh protsessov vyazkouprugikh ortotropnykh plastin s peremennoy zhestkostyu [Modelling of vibration processes of viscoelastic orthotropic plates with variable rigidity]. Mechanics. Proceedings of National Academy of Sciences of Armenia. 2011. No. 4(64). Pp. 30–38. (rus)
25. Eshmatov, B.Kh. Dinamicheskaya ustoychivost vyazkouprugikh plastin pri vozrastayushchikh szhimayushchikh nagruzkakh [Dynamic stability of viscoelastic plates under increasing compressing loads]. Journal of Applied Mechanics and Technical Physics. 2006. No. 2(47). Pp. 289–297. (rus)
26. Eshmatov, B.Kh. Nonlinear vibration analysis of viscoelastic plates based on a refined Timoshenko theory. Journal International Applies Mechanics. 2006. Vol. 42. No. 5. Pp. 596–605.
27. Eshmatov, B.Kh., Eshmatov, Kh., Khodzhaev, D.A. Nelineynyy flatter vyazkouprugikh pryamougolnykh plastin i tsilindricheskikh paneley iz kompozitsionnogo materiala s sosredotochennymi massami [Nonlinear flutter of viscoelastic rectangular plates and cylindrical panels of a composite with a concentrated masses]. Journal of Applied Mechanics and Technical Physics. 2013. No. 4(54). Pp. 578–587. (rus)
28. Abdikarimov, R.A., Zhgutov, V.M. Matematicheskiye modeli zadach nelineynoy dinamiki vyazkouprugikh ortotropnykh plastin i obolochek peremennoy tolshchiny [Mathematical models of problems of nonlinear dynamics of viscoelastic orthotropic plates and shells of variable thickness]. Magazine of Civil Engineering. 2010. No. 6. Pp. 38–47. (rus)
29. Chelomei, V.N. Dinamicheskaya ustoychivost elementov aviatsionnykh konstruktsiy [Dynamic stability of the elements of aircraft structures]. Moscow: Aeroflot, 1939. 80 p. (rus)
30. Ishmatov, A.N., Mirsaidov, M. Nonlinear Vibrations of an Axisymmetric Body Acted upon by Pulse Loads. Soviet Applied Mechanics. 1991. Vol. 27. No. 4. Pp. 68–74.
31. Sultanov, T.Z., Khodzhaev, D.A., Mirsaidov, M.M. Assessment of dynamic behavior of heterogeneous systems taking into account non–linear viscoelastic properties of soil. Magazine of Civil Engineering. 2014. No. 1(45). Pp. 80–89.
32. Mirsaidov, M., Mekhmonov, Ya. Non–axisymmetric vibrations of axisymmetric structures with associated mass and hollows (protrusions). J. Strength of Materials. 1987. No. 3. Pp. 111–116.
33. Verlan, A.F., Abdikarimov, R.A., Eshmatov, H. Chislennoye modelirovaniye nelineynykh zadach dinamiki vyazkouprugikh sistem s peremennoy zhestkostyu [Numerical modeling of nonlinear problems of the dynamics of viscoelastic systems with variable rigidit]. Electronic modeling. 2010. Vol. 32. No. 2. Pp. 3–14.
34. Koltunov, M.A., Kravchuk, A.S., Mayboroda, V.P. Prikladnaya mekhanika deformiruyemogo tverdogo tela

пластин переменной толщины // Журнал университета водных коммуникаций. 2011. № 2(9). С. 102–107.

24. Абдикаримов Р.А., Худаяров Б.А. Моделирование колебательных процессов вязкоупругих ортотропных пластин с переменной жёсткостью // Известия НАН Армении. Механика. 2011. № 4(64). С. 30–38.
25. Эшматов Б.Х. Динамическая устойчивость вязкоупругих пластин при возрастающих сжимающих нагрузках // Прикладная механика и техническая физика. 2006. Т. 47. № 2. С. 165–175.
26. Eshmatov B.Kh. Nonlinear vibration analysis of viscoelastic plates based on a refined Timoshenko theory // Journal International Applied Mechanics. 2006. Vol. 42. № 5. Pp. 596–605.
27. Эшматов Б.Х., Эшматов Х., Ходжаев Д.А. Нелинейный флаттер вязкоупругих прямоугольных пластин и цилиндрических панелей из композиционного материала с сосредоточенными массами // Прикладная механика и техническая физика. 2013. Т. 54. № 4. С. 74–85.
28. Абдикаримов Р.А., Жгутов В.М. Математические модели задач нелинейной динамики вязкоупругих ортотропных пластин и оболочек переменной толщины // Инженерно–строительный журнал. 2010. № 6. С. 38–47.
29. Челомей В.Н. Динамическая устойчивость элементов авиационных конструкций. М: Аэрофлот, 1939. 80 с.
30. Ishmatov A.N., Mirsaidov M. Nonlinear vibrations of an axisymmetric body acted upon by pulse loads // Soviet Applied Mechanics. 1991. Vol. 27. № 4. Pp. 68–74.
31. Султанов Т.З., Ходжаев Д.А., Мирсаидов М.М. Оценка динамического поведения неоднородных систем с учетом нелинейно-вязкоупругих свойств грунта // Инженерно-строительный журнал. 2014. № 1(45). С. 80–89
32. Mirsaidov M., Mekhmonov Ya. Non–axisymmetric vibrations of axisymmetric structures with associated masses and hollows (protrusions) // Journal Strength of Materials. 1987. № 3. Pp. 111–116.
33. Верлань А.Ф., Абдикаримов Р.А., Эшматов Х. Численное моделирование нелинейных задач динамики вязкоупругих систем с переменной жесткостью // Электронное моделирование. 2010. Т .32. № 2. С. 3–14.
34. Колтунов М.А., Кравчук А.С., Майборода В.П. Прикладная механика деформируемого твердого тела. М.: Высшая школа, 1983. 349 с.
35. Ржаницын А.Р. Теория ползучести. М.: Стройиздат, 1968. 416 с.

Mirsaidov, M.M., Abdikarimov, R.A., Vatin, N.I., Zhgutov, V.M., Khodzhaev, D.A., Normuminov, B.A. Nonlinear parametric oscillations of viscoelastic plate of variable thickness. Magazine of Civil Engineering. 2018. 82(6). Pp. 112–126. doi: 10.18720/MCE.82.11

[Applied mechanics of a deformable solid]. Moscow: High School. 1983. 349 p.

35. Rzhanitsyn, A.R. Teoriya polzuchesti [Creep Theory]. Moscow: Stroiizdat, 1968. 416 p.

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