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Spatial stress state and dynamic characteristics of earth dams

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Abstract. Strength assessment of earth dams is mainly conducted using a plane design scheme, which does not always lead to adequate results. In this paper, it is proposed to assess the stress state of earth dams in a three-dimensional statement. Consequently, to assess the stress-strain state and dynamic characteristics of earth dams, appropriate mathematical models, methods and algorithms are built. The basis of the developed methods for solving specific problems for a spatial structure is a finite element method, the Gauss method (or the square root method) and the Muller method. Reliability of results is proved by solving a series of test problems. With the developed methods, the stress-strain state and dynamic characteristics of the Gissarak and Sokh earth dams were investigated. Based on the results of the study, it has been shown that for some types of earth dams, at preliminary assessment of the stress state and dynamic characteristics of structures, it is possible to use a plane-deformable model of calculation. Studies have shown that to ensure the required accuracy in assessing the stress state and dynamic characteristics of complex inhomogeneous spatial systems (such as earth dams), it is necessary to make calculations using a three-dimensional model. The data obtained as a result of research allowed to reveal some features of the stress state in a spatial case, indicating dangerous areas with the greatest stresses, as well as to study the pattern of natural oscillations that cannot be described using a plane model.

1. Introduction

Correct definition of the stress-strain state (SSS) and dynamic characteristics of the object under consideration is a major factor in assessing the strength of structures. Reliable definition of these parameters, in turn, depends on the chosen design scheme of structure, used mathematical models describing the processes occurring in the object under consideration, the equations of material state and the solution methods of considered problems [1–5].

Recently several papers [6–14] have been published, where static and dynamic stress-strain states of various earth dams are considered in plane and spatial statements, taking into account various factors, such as design features of structure, moisture-content properties of soil, structure interaction with water reservoir and hydro-mechanical phenomena.

Along with that, it is necessary to mention separately the following papers devoted to the solution of various topical issues related to the state assessment of earth structures.

In [10], the state of the dam was analyzed by numerical simulation taking into account water-saturated soil and hydro-mechanical phenomena. Material selection for the design of the dam was discussed.

Using the ABAQUS software package the state of the dam was analyzed in [11], taking into account the interaction of the dam with water of the reservoir. Obtained results have shown that the neglect of this factor leads to an overestimated value of stability and, as a result, to structure damage. The effect of clay and rock bases on the cracks propagation in the dam body was considered.

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Reliability indices and safety factors of dams [12] were evaluated for various heights and slopes. It was stated that the values of safety factors for high dams in normal conditions should be no less than 1.70 and in seismic conditions no less than 1.40.

In [13, 14], the effect of hydrodynamic pressure on dam state during earthquakes was considered in a plane statement. The results obtained showed that the neglect of rigid body-water interaction effect led not only to an overestimation of the acceleration reaction in the rock-fill material, but also to an overestimation of dynamic stresses in a structure.

Dynamic characteristics of the dam with filled reservoir were studied experimentally and numerically in [14]. It was stated that the acceleration gain factors varied with the height of the dam depending on natural frequency, modes of structure oscillations, depth of the reservoir, and other factors. It was recommended to take these factors into consideration when designing a structure.

In [15], static and dynamic elastic-plastic analysis of the dam state by the finite element method was carried out during the Wenchuan earthquake. To describe the properties of rock-fill material, an elastoplastic model was used, taking into account the destruction of particles. Numerical calculations obtained corresponded to a great extent to field measurements during construction and after the Wenchuan earthquake.

Bending strain of the dam was investigated in [16], since bending often led to structure destruction. Using special programs and the finite element method, numerical modeling of dam bending rate was carried out. The results of numerical simulation and statistical analysis have shown that an increase in elastic modulus, Poisson’s ratio, internal friction angle and the ratio of core thickness to filter thickness would result in a decrease in bending.

Hydro-mechanical properties of traditional and unconventional materials (i.e. clay material, masonry) used in the construction of dams were analyzed in [17]. The use of unconventional material (soil and stone mixtures) for reasons of ensuring dam stability was analyzed in more detail.

The above review of published papers shows that the problem of earth dam spatial calculation is studied insufficiently [18–24] and therefore is of great interest.

Usually, when evaluating the stress-strain state of dams located in wide dam sites, it is sufficient to use plane strain conditions, however, the use of plane design schemes for earth dams located in narrow dam sites still requires more careful checking of the accuracy of the SSS estimation of structures.

The state of earth dam under various effects is also determined by its length. As noted in [25, 26] for extended dams (with a ratio of the crest length — L_{cr} to the dam height H equal to $L_{cr}/H \geq 6$), calculations can be made according to the plane strain scheme; in this case, it is possible to assess not the entire structure, but only its central section. If the given ratio is violated, then the spatial nature of the dam is revealed. At the same time, as studies in [6, 18, 19, 24–26] show, the accuracy of dam calculation changes not only under static load, but under dynamic effect as well.

Brief review presented here shows that obtaining of reliable results in the SSS calculation and assessment of dynamic characteristics of earth dams are quite serious problems, since the task to develop sound quantitative estimates of structure strength with account of actual geometric dimensions of earth dams dictates the need to take into account spatial nature of structure operation [4, 6, 7].

Therefore, at present, it is necessary to give primary recommendations to assess the stress-strain state and dynamic characteristics of earth dams by adequate design models that describe the actual features of a structure.

This paper is devoted to the solution of the following issues:

– to assess the stress-strain state and dynamic characteristics of earth dams, a three-dimensional (spatial) model is proposed that takes into account inhomogeneous and geometric features of a structure;

– a variation statement of the problem is given taking into account spatial strain state of earth dams under consideration;

– the methods to solve specific tasks for real structures using spatial finite elements are proposed;

– the stress-strain state and dynamic characteristics of real earth dams are studied using spatial (that is, three-dimensional) models and the models of plane strain;

– plane (plane-strain) and three-dimensional (spatial) models;

– primary conclusions on the use of three-dimensional models in assessment of the SSS and dynamic characteristics of specific earth dams are given, based on the analysis of the results obtained.

2. Methods

Dynamic characteristics, that is, natural frequencies, oscillation modes and damping coefficients of structures are determined by studying natural oscillations of structures. Dynamic characteristics of a structure are the passport of the considered structure, allowing evaluating in advance dynamic properties of a structure as a whole.

2.1. Mathematical model

To assess the stress-strain state (SSS) and dynamic characteristics of earth dams, an inhomogeneous three-dimensional model of the system, Figure 1, is considered, in which the base surfaces and side slopes $\Sigma_0, \Sigma_{S_1}, \Sigma_{S_2}$ are rigidly fixed, the surface of the downstream Σ_3 is stress-free, the hydrostatic pressure of water acts on surface Σ_1 (on the part of upstream slope which is lower than the BSL line, *Sp*), and an external load is applied to a part of crest surface Σ_2 of the site Σ_p.

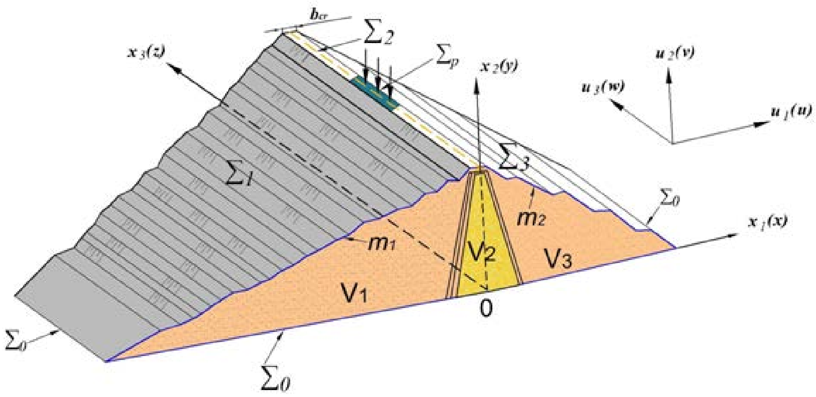

Figure 1. Three-dimensional model of inhomogeneous system.

Here $V = V_1 + V_2 + V_3$ is the capacity of the dam body (V_1, V_3 are the capacities of the upper and lower retaining prisms, V_2 is the capacity of the core); $\Sigma_{S_1}, \Sigma_{S_2}$ are the surfaces of the coastal slopes, Σ_0 is the surface of the base along the bottom, and Σ_1, Σ_2, Σ_3 are the surfaces of the retaining prisms and the crest.

System material is considered to be elastic. In calculations the mass forces $\vec{f}$ acting on the system and hydrostatic pressure of water $\vec{p}$ are taken into account [4].

The aim is to determine the fields of displacements, stresses and dynamic characteristics of an earth dam of $V = V_1 + V_2 + V_3$ capacity (Figure 1).

To simulate the process of strain and to assess dynamic characteristics of earth dams (Figure 1) in a three-dimensional statement, the Lagrange variation equation, based on the d'Alembert principle for inhomogeneous deformable bodies is used:

$$\begin{gathered}
-\int_{V_1} \sigma_{ij}\delta\varepsilon_{ij} dV - \int_{V_2} \sigma_{ij}\delta\varepsilon_{ij} dV - \int_{V_3} \sigma_{ij}\delta\varepsilon_{ij} dV - \\
-\int_{V_1} \rho_1 \ddot{\vec{u}}\, \delta\vec{u}\, dV - \int_{V_2} \rho_2 \ddot{\vec{u}}\, \delta\vec{u}\, dV - \int_{V_3} \rho_3 \ddot{\vec{u}}\, \delta\vec{u}\, dV + \\
+\int_{V} \vec{f}\, \delta\vec{u} dV + \int_{S_P} \vec{p}\, \delta\vec{u} dS + \int_{\Sigma_P} \vec{P}_1\, \delta\vec{u} d\Sigma = 0.
\end{gathered} \tag{1}$$

Physical properties of system material are described by the relations between stresses σ_{ij} and strains ε_{ij} in the form [27]:

$$\sigma_{ij} = \lambda_n \varepsilon_{kk} \delta_{ij} + 2\mu_n \varepsilon_{ij} \tag{2}$$

the relationship between the components of the strain tensor and the displacement vector is described by the linear Cauchy relations [27]

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \quad (3)$$

Further, when building mathematical models, uniform kinematic boundary conditions are taken into account:

$$\vec{x} \in \sum\nolimits_0 : \vec{u} = 0. \quad (4)$$

Here $\vec{u}$, ε_{ij}, σ_{ij} are the displacement vector, the strain and stress tensors, respectively;

$\delta\vec{u}$, $\delta\varepsilon_{ij}$ are the isochronous variations of displacements and strains;

ρ_n is the material density of the system elements under consideration (index n = 1, 2, 3 means different parts of the dam to which this quantity relates);

$\vec{f}$ is the vector of mass forces;

$\vec{P}_1$ is the vector of external forces applied to area Σ_p;

$\vec{p}$ is the hydrostatic pressure of water.

In all the problems considered, the displacement vector in spatial coordinate system $\vec{x} = \{x_1, x_2, x_3\} = \{x, y, z\}$ has three components $\vec{u} = \{u_1, u_2, u_3\} = \{u, v, w\}$ in all relations i, j, k = 1, 2, 3.

Now, the variation problem to assess the stress-strain state of an earth dam can be formulated as follows: it is necessary to determine (without taking into account the inertial forces) fields of displacements $\vec{u}(\vec{x}, t)$, strains $\varepsilon_{ij}(\vec{x}, t)$ and stresses $\sigma_{ij}(\vec{x}, t)$ in an inhomogeneous three-dimensional system (Figure 1) arising under mass forces ($\vec{f}$), external forces ($\vec{P}_1$) and hydrostatic pressure of water ($\vec{p}$), satisfying equations (1), (2), (3) and corresponding to kinematic conditions (4) at any possible displacement $\delta\vec{u}$.

In the case of determining dynamic characteristics of an earth dam, the variation problem under consideration can be formulated as follows: it is necessary to determine the most ordered movements of the system point, occurring according to a harmonic law at different amplitudes in the absence of external influences, i.e. ($\vec{f}$), ($\vec{P}_1$), ($\vec{p}$), satisfying the equations (1), (2), (3) and corresponding to boundary conditions (4) at any possible displacement $\delta\vec{u}$.

2.2. Method and algorithm

Variation problem set above (1)–(4) of the SSS assessment of inhomogeneous systems (Figure 1), under the effect of hydrostatic pressure and mass forces, taking into account spatial factors, with the use of a finite element (in the form of a hexahedral parallelepiped with 24 degrees of freedom) is reduced to the resolving system of algebraic equations of N-th order:

$$[K]\{u\} = \{P\}, \quad (5)$$

where the element stiffness matrix $[K]$ of the system (Figure 1) is constant and depends on elastic physicomechanical parameters of the system;

$\{u\}$ is the sought for vector of nodal displacements;

$\{P\}$ is the vector of external load (mass forces and hydrostatic pressure of water).

When deriving equation (5), the stiffness matrix $[K]$ and nodal forces $\{P\}$ are formed automatically using the algorithm given in [4, 20].

Kinematic boundary conditions (4) are taken into account when forming the system of equations (5), restricting its order only to equations that do not contain zero displacements. The order of the formed systems of algebraic equations (5) in some calculations exceeded 4000.

Solution of the obtained system of algebraic equations (5) was performed by the Gauss method and the square root method [29, 30], taking into account the tape structure of the stiffness matrix. Solving the system of equations (5), the components of displacements (u_1, u_2, u_3) at each point of the system (i.e. displacement fields) are determined; then using these data the components of strain (3) and stress (2) tensors are determined, taking into account inhomogeneous structural features of systems.

When determining the natural frequencies and modes of oscillation, the considered variation problem (1)–(4), with the use of a finite element (in the form of a hexahedral parallelepiped with 24 degrees of freedom), is reduced to an algebraic eigenvalue problem for a homogeneous system of algebraic equations of *N*-th order:

$$\{[K]-\omega^2[M]\}\{u\}=0. \tag{6}$$

Here $[K]$, $[M]$ are the stiffness and mass matrices, and ω, $\{u\}$ are the sought for eigenfrequency and eigenvector of the system under consideration (Figure 1). The elements of stiffness matrix $[K]$ are constant and depend only on elastic physicomechanical parameters of structure material.

Solution of equations (6), (i.e. determination of eigenfrequencies) is carried out by the Muller method [4, 31], and determination of eigenvector – by the Gauss method or the square root method [29, 30].

The order of the systems of algebraic equations (6) in some calculations reached 3000.

The algorithm to determine the eigenfrequencies and vectors of algebraic problems on eigenvalue is described in detail in [4]. The essence of this algorithm is as follows:

1. With the iterative procedure of the Muller method, a sequence of eigenvalues of algebraic equations (6) $\lambda_1, \lambda_2, \ldots, \lambda_n$ is determined.

2. Then, using expression ($\lambda = \omega^2$), the eigenfrequencies $\omega_1, \omega_2, ..., \omega_n$ of algebraic equations (6) are calculated.

3. By substituting the found values $\omega_1, \omega_2, ..., \omega_n$ in equation (6), eigenvectors in (6) (i.e., oscillation modes) $\{u_1\}, \{u_2\}, ..., \{u_n\}$ are obtained by the Gauss method or the square root method [29, 30].

The program for determining eigenfrequencies and modes of oscillations of inhomogeneous spatial systems is protected by the copyright certificate of the Patent Agency of the Republic of Uzbekistan.

2.3. Test problems to check the accuracy of the methods and computing algorithms

This section verifies the accuracy of the developed methods and algorithms solving test problems for which the exact or numerical solution is known.

Problem 1. Consider elastic spatial structure in the form of a long rectangular parallelepiped under uniform pressure P acting on the upper surface.

The parallelepiped rests on an absolutely rigid and smooth base, i.e.

$$x_2 = 0: u_2 = 0; \sigma_{12} = 0; \ \sigma_{13} = 0. \tag{7}$$

Surface load is applied on the upper surface in the form of uniform pressure

$$x_2 = a: \ \sigma_{22} = -P. \tag{8}$$

The side surface of the parallelepiped is stress-free.

It is required to determine displacements and stresses at various points of the parallelepiped under pressure P using plane-deformable and spatial models. For the plane problem, the exact solution is known [34]. In a specific calculation, the following geometrical parameters of the parallelepiped and mechanical characteristics of material were used: P = 1 tf/m^2; cross-sectional dimensions $a = b$ = 2.0 m; the modulus of material elasticity E = 1.0 tf/m^2 and the Poisson's ratio μ = 0.3.

Comparison of exact results (Table 1) with numerical solutions (Table 2) for the same points of the parallelepiped shows good agreement of the values obtained both in terms of the displacement components and in stress tensor components.

Problem 2. Natural oscillations of a body (height H = 8.0 m) of rectangular cross sections (b = 0.5 m; h = 0.5 m), rigidly fixed along the base ($x_2 = 0$) and with free upper end ($x_2 = H$) are considered in the problem.

Table 1. Exact solution of plane problem for section (x_3 = 0.0 m) of the parallelepiped.

Coordinates, x_2 (m)	x_1 = –1.0 m					x_1 = 1.0 m				
	u_1	u_2	σ_{11}	σ_{22}	σ_{12}	u_1	u_2	σ_{11}	σ_{22}	σ_{12}
0.0	–0.39	0.0	0	–1	0	0.39	0.0	0	–1	0
0.5	–0.39	–0.455	0	–1	0	0.39	–0.455	0	–1	0
1.0	–0.39	–0.910	0	–1	0	0.39	–0.910	0	–1	0
1.5	–0.39	–1.365	0	–1	0	0.39	–1.365	0	–1	0
2.0	–0.39	–1.820	0	–1	0	0.39	–1.820	0	–1	0

Table 2. Solution of spatial problem for section (x_3 = 0.0 m) of the parallelepiped.

Coordinate, x_2 (m)	x_1 = –1.0 m								
	u_1	u_2	u_3	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{13}	σ_{23}
0.0	–0.3889	0.0	–0.0	–0.2996	–0.9983	$-0.41e^{-3}$	$0.131e^{-12}$	$0.157e^{-10}$	$0.797e^{-4}$
0.5	–0.3886	–0.4541	$-0.417e^{-14}$	–0.2966	–0.9980	$0.258e^{-3}$	$0.202e^{-14}$	$0.246e^{-11}$	$0.600e^{-4}$
1.0	–0.3883	–0.9091	$-0.617e^{-14}$	–0.2947	–0.9989	$0.245e^{-3}$	$-0.139e^{-13}$	$0.629e^{-11}$	$0.235e^{-4}$
1.5	–0.3881	–1.3648	$-0.727e^{-14}$	–0.2935	–0.9994	$0.257e^{-3}$	$-0.122e^{-13}$	$0.509e^{-11}$	$0.360e^{-6}$
2.0	–0.3880	–1.8207	$-0.947e^{-14}$	–0.2930	–0.9996	$0.227e^{-3}$	$0.556e^{-14}$	$0.29e^{-11}$	$-0.607e^{-15}$

Natural frequencies and oscillation modes of this body are determined using one-dimensional and spatial models.

In one-dimensional statement, this problem has an exact solution [35] for the eigenfrequencies ω_i.

Solving this problem, the following mechanical parameters of material were used: specific weight of material γ = 1.0 tf/m^3; elastic modulus E = 1.0 tf/m^2 and Poisson's ratio μ = 0.25.

Exact solution of the body, obtained by one-dimensional model is compared in Table 3 with numerical results obtained by three-dimensional models (using spatial finite elements) with developed computer program.

Table 3. Table of eigenfrequencies.

Number of eigenfrequency	Eigenfrequencies of a body, rad/sec			
	Exact solution by one-dimensional model			Numerical solution obtained by the FEM with three-dimensional model
	Bending	Longitudinal	Torsional	
1	2	3	4	5
1	**0.0248***	**0.6150****	0.1459	**0.0248***
2	**0.1556***	**1.8450****	0.4376	**0.1531***
3	0.4358	3.0749	0.7294	0.3536
4	0.8494	4.3049	1.0211	0.4174
5	1.4117	5.5349	1.3129	**0.6159****
6	2.1089	6.7648	1.6047	0.7895
7	2.9455	7.9948	1.8964	1.0606
8	3.9215	9.2248	2.1882	1.2521
9	5.0370	10.4550	**2.4799*****	1.2522
10	6.2919	11.6851	2.7717	1.7670
11				1.7869
12				**1.8447****
13				2.3778
14				2.3779
15				**2.4725*****

In Table 3 one asterisk indicates the frequencies corresponding to the bending modes of oscillations, two asterisks — the longitudinal modes and three asterisks — the torsional modes of the body, obtained by spatial model.

Comparison of the exact value of natural frequencies of one-dimensional model with numerical values obtained by the FEM with spatial model

Partial coincidence of corresponding frequencies with results of one-dimensional model exact solution and with results obtained using the developed computer programs with three-dimensional model proves the reliability of the developed methods, algorithms and the calculation program when determining natural frequencies and oscillation modes of a spatial body.

Analyzing the obtained eigenmodes of body oscillations using spatial model, the spatial nature should be noted: only the first two are the bending modes, the fifth is a longitudinal mode, and in the others, that is, in the 3rd, 4th, 6th, 7th, 8th and 9th modes of oscillations the strain has a spatial character. This shows that one-dimensional model is not able to fully describe the real strain under natural oscillations even of a thin bar.

3. Results and Discussion

Using the developed methods and algorithms the stress-strain state of Gissarak and Sokh earth dams are studied in spatial statement, taking into account the actual geometric dimensions of the structure and uniform and non-uniform structural features of a dam, arising under gravitational forces (own weight) and hydrostatic pressure of water.

3.1. Assessment of the stress-strain state

As specific examples we have considered

– **the Gissarak earth dam**: height H = 138.5 m*;* base width B_w = 634.0 m; crest width b_{cr} = 16.0 m; inclination coefficients of upstream and downstream slopes m_1 = 2.2, m_2 = 1.9; longitudinal dimensions: base length L_b = 140.0 m, crest length L_{cr} = 660.0 m. Mechanical characteristics of material of different sections of the dam are: volume weight γ — kgf/cm^3, Poisson's ratio — μ , shear modulus G — kgf/cm^2: core — γ = 0.0017 kgf/cm^3, μ = 0.37, G = 2780 kgf/cm^2; transition zones — γ = 0.00215 kgf/cm^3*,* μ = 0.35, G = 3500 kgf/cm^2; retaining prisms — γ = 0.0024 kgf/cm^3, μ = 0.25, G = 3210 kgf/cm^2; slope strengthening — γ = 0.0017 kgf/cm^3, μ = 0.37, G = 84000 kgf/cm^2;

– **the Sokh earth dam:** height H = 87.3 m: base width B_w = 530.0 m; crest width b_{cr} = 10.0 m; inclination coefficients of upstream and downstream slopes m_1 = 2.5, m_2 = 2.2; longitudinal dimensions: base length L_b = 21.0 m, crest length L_{cr} = 48.75 m. Mechanical characteristics of material of different sections of the dam are: volume weight — γ, Poisson's ratio — μ*,* shear modulus — G: core — γ = 0.0017 kgf/cm^3, μ = 0.40, G = 2820 kgf/cm^2 retaining prisms — γ = 0.0021 kgf/cm^3, μ = 0.35*,* G = 3160 kgf/cm^2; slope strengthening — γ = 0.00185 kgf/cm^3, μ = 0.35*,* G = 3100 kgf/cm^2.

Figure 2 shows the isolines of the stress tensor components distribution for the central section of the Gissarak (Figure 2a, b, c) and Sokh (Figure 2d, e, f) dams under their own weight, obtained using a spatial model of structures. Stress magnitudes in all figures are indicated in MPa.

Coordinate axes shown in Figure 1 have the following directions: the axes x_1 (horizontal) and x_2 (vertical) are in the plane of the central section; the x_3 axis is perpendicular to this plane.

Comparison of the results obtained using spatial model (Figure 2a, b, c) for the Gissarak dam with plane-deformable model shows almost the same pattern of stress distribution and their close values, i.e. under vertical gravitational load acting on the dam, the patterns of stress components distribution (σ_{11}, σ_{12}, σ_{22}) in the central section of the dam are identical. The maximum values of horizontal σ_{11} and vertical σ_{22} stresses are observed at the bottom of the central — the highest — part of the dam (Figure 2a, b), and shear stresses σ_{12} — along the slopes (Figure 2c). At the same time, the distribution of stresses (Figure 2a, b, c) relative to the x_2 axis is almost symmetrical, since the structure itself is also almost symmetrical with a slight difference in inclination coefficients of the slopes (m_1 = 2.2, m_2 = 1.9).

When obtaining results in the vicinity of the onshore zones of the structure in contact with mountain slopes, the condition of rigid fixation was set. The results obtained with spatial model have no plane analogues. Therefore, when calculating this zone, it is necessary to use a spatial model.

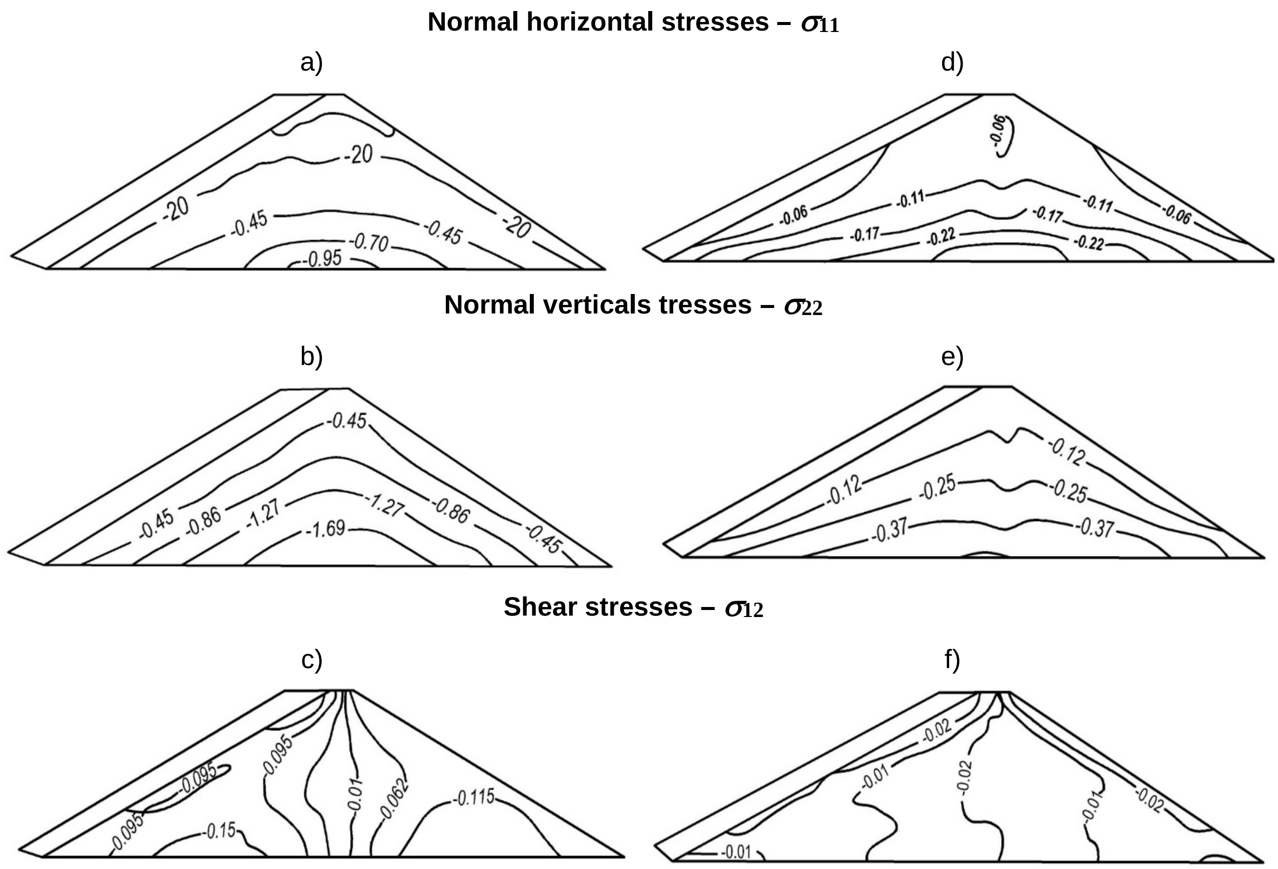

Figure 2. Isolines of distribution of stress tensor components in the central section of the Gissarak (a, b, c) and the Sokh (d, e, f) dams obtained using spatial model under own weight.

The ratios of geometric dimensions for the Gissarak dam (the length of the crest to the height — L_{cr}/H, the length of the base to the height — L_0/H and the width of the base to the height — B_w/H) are: L_{cr}/H = 4.78, L_0/H = 1.01, B_w/H = 4.57, respectively. Therefore, the use of plane and spatial models for this dam gives almost identical results, so, it is possible to use plane models in static calculations of such dams.

Isolines of stress components distribution of the Sokh dam, obtained with spatial model (Figure 2d, e, f) shows a significant difference from the results obtained using plane-deformable models. The maximum values of normal horizontal stresses – σ_{11} and vertical stresses – σ_{22} (Figure 2c, d) occur at the bottom of the central – the highest – part of the dam, and shear stresses σ_{12} – along the slopes (Figure 2e) and in upstream retaining prism. The analysis of the results obtained shows that the calculation of this dam using a plane-deformable model does not provide necessary accuracy at estimatingthe stress state of the dams of such dimensions. This, apparently, is explained by the small ratios of the horizontal dimensions of the dam to its height, which are L_{cr}/H = 0.56, L_0/H = 0.24, B_w/H = 6.07. In this case, the structure is not extended and does not meet the condition to choose a model of plane strain. It should be considered as a three-dimensional body.

The analysis of stress state of the Gissarak dam under its own weight, by plane and spatial models, shows a qualitative and quantitative identity of stresses across the section.

Thus, the stress state of dams, geometrically similar to the Gissarak dam under gravitation forces, with satisfactory accuracy can be described by a model of plane strain. As for the evaluation of stresses in the plane of dam site, the study is possible with a spatial model, while the stress state of the Sokh dam and the ones similar in geometry, must be estimated with a spatial model.

Analysis of the stress state of the above dams has shown that the maximum shear stresses σ_{23} under vertical gravitational load arise along the lateral boundaries of the dam site and can cause shear and cracks at the sides. Vertical normal stresses σ_{22} occur in the central part of the dam base. Maximal (in the modulus) horizontal stresses σ_{11} appear at the base along the crest; their negative values in the central part indicate the compression of the central part under the crest, and positive values indicate the heaving of the lower side of the side slopes at the dam body settlement under its own weight. Positive values of horizontal stresses σ_{33} in the upper part of the sides indicate the possibility of fractures and cracks on the side slopes.

Thus, a spatial model allows estimating the stress-strain state not only in the cross section, but also in the dam junction with the gorge, where the danger zones appear.

Further, the possibility of using the models of plane strain for solving specific problems for real dams should be proved by studying the stress state, taking into account geometric dimensions, structural features and spatial factors of the structures under consideration.

3.2. Assessment of dynamic characteristics of a dam

Spatial eigenfrequencies and oscillation modes of earth dams considered above, have been studied further, taking into account the non-uniform structural features.

Figure 3 shows natural modes of oscillations of the Gissarak dam, corresponding to the eigenfrequencies obtained.

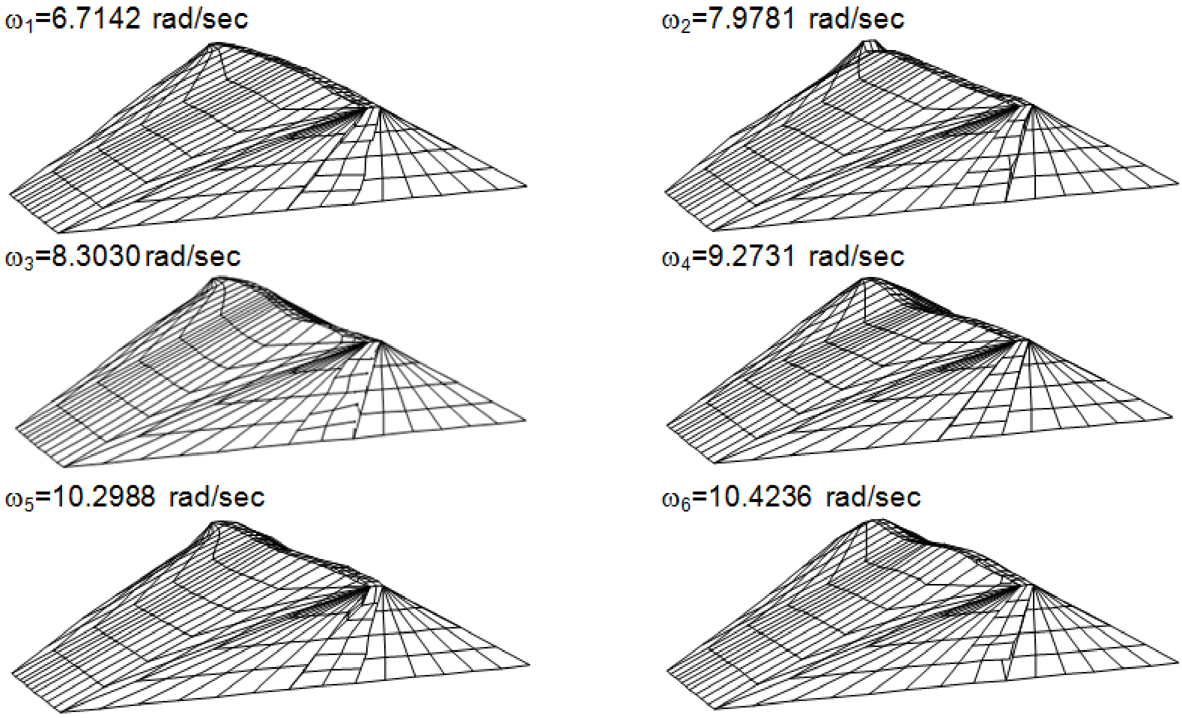

Figure 3. Natural modes of oscillations of the Gissarak dam, obtained using the spatial model.

An analysis of spatial eigenfrequencies distribution of the dam indicates the existence of more dense spectrum in a wider range, that is: ω_1 = 6.7142 rad/sec; ω_2 = 7.9781 rad/sec; ω_3 = 8.3030 rad/sec; ω_4 = 9.2731 rad/sec; ω_5 = 10.2988 rad/sec; ω_6 = 10.4236 rad/sec; ω_7 = 10.8966 rad/sec; ω_8 = 11.0334 rad/sec; ω_9 = 11.1865 rad/sec; ω_{10} = 11.8026 rad/sec; ω_{11} = 12.0947 rad/sec; ω_{12} = 12.3554 rad/sec; ω_{13} = 12.4357 rad/sec; ω_{14} = 12.7373 rad/sec; ω_{15} = 12.9691 rad/sec; ω_{16} = 13.1138 rad/sec; ω_{17} = 13.3937 rad/sec; ω_{18} = 13.4548 rad/sec; ω_{19} = 14.2052 rad/sec; ω_{20} = 14.3668 rad/sec.

This is explained by the fact that different models have different number of degrees of freedom, each of which makes an additional contribution to the spectrum of fundamental frequencies of natural vibrations. The frequencies obtained for this dam, reflecting shear and vertical oscillations of the central cross section, are almost identical in spatial and plane models.

For a spatial model, the shift of the central cross section is the bending of the longitudinal axis of the model (x_3). Subsequent oscillation frequencies of spatial model are the highest forms of bending of longitudinal axis (x_3), not considered by plane model. Therefore, the frequency spectrum in spatial case is denser, since between the main frequencies there are intermediate ones, reflecting higher modes of bending vibrations of the longitudinal axis of model (x_3).

As for the pattern of oscillations modes, we can note the following. Fundamental modes, reflected by spatial and plane models, are: the shear of the central section (the first mode); vertical displacements of the central section (for the plane model this is the second mode); complex deformations of the central section slope (for the plane model, this is the third and subsequent modes). For a spatial model, all modes, including mentioned above, are accompanied by a bending in longitudinal axis of the model (a crest) along the horizontal and vertical axes. For the above modes – these are the main modes of bending, for the subsequent ones – this is a bending with nodes. The bending of the crest (main one and with nodes) is accompanied by complex deformations of the structure slopes, and not only in their central part (central section), but also over the entire surface. These additional modes, and corresponding frequencies, are not reflected by plane model but fully reflected by spatial one. However, its use, as noted above, is quite laborious. The choice of a particular model when considering specific structures should be substantiated not only by the geometry of the object, but also by the estimated load, its direction and frequency spectrum, which can cause structure oscillations not only of fundamental mode but also in higher modes not reflected by plane model [32, 33].

The question of how great the longitudinal strain along the axis (x_3) is, remains the priority, since the validity of using a plane model depends on it. If longitudinal displacements under a certain effect are large, then in this case the use of a plane model is unacceptable, since the possibility of transverse cracks formation is not taken into account.

To answer the question, it is necessary to consider the problem of unsteady forced oscillations of the dam under different (in directions) kinematic effects using plane and spatial models and to compare the results obtained.

4. Conclusions

1**.** A mathematical model, methods and algorithm for estimating the stress-strain state and dynamic characteristics of inhomogeneous spatial systems using a spatial model are presented.

2. The stress-strain state and dynamic characteristics of two different earth dams using three-dimensional models is estimated taking into account actual geometric dimensions and inhomogeneous features of a structure.

3. An analysis of obtained results on the stress-strain state assessment of earth dams with spatial models has shown that for some types of dams it is possible to use plane strained models to obtain results with acceptable accuracy. Nevertheless, in each case, for specific structures, in assessing the stress state of dams, it is necessary to check the stress state using a spatial model.

4. The use of spatial model makes it possible to identify dangerous zones of the structure (where higher stresses occur compared to other areas), which could not be identified using a plane model.

5. Analysis of dynamic characteristics of dams with a plane model has revealed a rather dense spectrum of spatial eigenfrequencies and the identity of fundamental modes of natural oscillations over the cross section of a dam.

6. The first three fundamental modes for the Gissarak dam are: the displacement of the central section (the first mode); vertical compression of the section (the second mode); complex deformations of the slopes of central section (the third mode), etc.

7. For a spatial model, all modes are accompanied by a bending in longitudinal axis (a crest) in different planes. The bending of the crest is accompanied by complex deformations of the structure slopes, not reflected by plane model. The eigenfrequencies corresponding to these modes make an additional contribution to the frequency spectrum, condensing and expanding its range.

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Пространственное напряженное состояние и динамические характеристики грунтовых плотин

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Ключевые слова: пространственная система, трехмерная (пространственная) модель, неоднородность, грунтовая плотина, напряженно-деформированное состояние, динамическая характеристика, собственная частота и форма колебаний

Аннотация. Оценка прочности грунтовых плотин, в основном, производится с использованием плоской расчетной схемы, которая не всегда приводит к адекватным результатам. В данной работе предлагается провести оценку напряженного состояния грунтовых плотин в трехмерной постановке. Следовательно, для оценки напряженно-деформированного состояния и динамических характеристик грунтовых плотин строятся соответствующие математические модели, методика и алгоритм. В основу разработанной методики при решении конкретных задач для пространственного сооружения заложен метод конечных элементов, метод Гаусса (или метод квадратного корня) и метод Мюллера. Достоверность полученных результатов проверена решением ряда тестовых задач. С помощью разработанной методики исследованы напряженно-деформированное состояние и динамические характеристики Гиссаракской и Сохской грунтовых плотин. На основе результатов исследования показано, что для некоторых типов грунтовых плотин, при предварительной оценке напряженного состояния и динамических характеристик сооружений, возможно использование плоско-деформируемой модели расчета. Проведенные исследования показали, что для обеспечения требуемой точности при оценке напряженного состояния и динамических характеристик сложных неоднородных пространственных систем (т.е. грунтовых плотин) необходимо проводить расчеты с использованием трехмерной модели. Полученные в результате исследований данные позволили выявить некоторые особенности напряженного состояния в пространственном случае, указывающие на возникающие опасные участки с наибольшими напряжениями, а также изучить характер собственных колебаний, которые невозможно описать использованием плоской модели.

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