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Mixed finite-element method in V.I. Slivker's semi-shear thin-walled bar theory

V.V. Lalin, V.A. Rybakov*, S.S. Ivanov, A.A. Azarov
Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russia
** E-mail: fishermanoff@mail.ru*

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Abstract. Mixed variational formulation of static and dynamic problems for thin-walled beams is presented. Stiffness and mass matrixes are derived from the Reissner-like functional. Shear deformation is taken into account by using Slivker's semi-shear theory of thin-walled bars. Corresponding Euler equations are derived from the proposed mixed functional. Linear Hermite polynomials were considered as approximation for all the internal forces and displacements functions. The exact analytical solutions to some particular eigenfrequency and static problems for thin-walled beam are obtained from mixed formulation. The effect of "spurious" frequencies in thin-walled beam spectrum is discussed. Comparison of the numerical results from the mixed and classical finite element methods is presented.

1. Introduction

Classical finite element method (FEM) applied to structural mechanics has a serious drawback concerning accuracy of stress calculation. As displacements are the only unknown functions approximated by some polynomials, the process of evaluating stresses appears to be secondary to the main problem of solving system of linear equations. Due to the necessity of taking derivative, the degree of polynomials forming the Ritz sum in stress expression decreases which means that calculation precision suffers.

Thin-walled beam, because of its specific properties, has a unique internal force factor — bimoment, which in certain cases can cause large normal stresses in cross-section. In order to determine that stresses with good accuracy mixed FEM can be used.

The first mixed formulation in classic calculus of variations was presented by Hellinger. However, only after the work [1] by Reissner the mixed variational principle has become a common tool for variational formulations of problems in structural mechanics. The main property of the Reissner's functional is that it includes stress functions along with displacements, which gives an advantage of choosing the approximation functions for every unknown value independently.

The first theory of thin-walled beams of open cross-section was developed by Vlasov in [2], and since that work many researches have been devoted to the application of this theory to a variety of static and dynamic problems of thin-walled structures. The problem of free vibrations of thin-walled beam with open cross-section was solved in [3]. In this article, based on the solution of ordinary differential equations governing the static problem, stiffness and consistent mass matrixes were derived. In addition, some numerical experiments were carried out in order to prove sufficient accuracy of the lumped mass matrix formulation. In [4] another finite element model, based on Vlasov's theory, was introduced. In this work special attention was paid to the computational problems arising when coefficients of stiffness matrix vanish whilst compiling the global governing system of equations. The influence of boundary conditions on coupled vibrations of thin-walled bar was investigated. Galerkin's method was used for spectrum analysis in article [5], which showed sufficient accuracy of results even for the problems that do not have analytical solution. An anisotropic

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composite material of thin-walled bar considered by author has led to introduction of nonclassical parameters such as warping restraint, transverse shear flexibility and structural couplings.

In [6] author clarified Vlasov's equations of motion for thin-walled beams taking into account torsional shear deformation in middle surface. In addition, that paper introduced some experiments on thin-walled beam resonant frequency and verification with the theoretical results. Simply supported thin beams were numerically and empirically investigated in [7]. This work presented so-called "engineering" theory of bars based on analogy with elastic behavior of thin plates. Due to its relative simplicity, this theory could be applied in designing practice, although it is not valid for other boundary conditions. The method of trigonometric series used in this article to approximate stress function is suitable for finite element analysis as well.

The expressions describing thin-walled rod vibrations in [8] were presented as four partial, linear integro-differential equations. The complex formulation allowed to take into account the effect of longitudinal inertia and shear flexibility. Moreover, the application area of the generalized theory is not limited to open cross-section beams with uniform mechanical and geometrical properties. Vlasov's and Timoshenko's equations of motion for thin-walled beam appear to be special cases of this theory if relevant hypotheses are accepted.

The influence of shear deformation and rotational degree of freedom on thin-walled beam vibrations were considered in [9]. The governing equations were derived from Vlasov's theory but with some additional modifications concerning shear ductility. One advantage of the proposed theory is that it allows obtaining dynamic characteristics of beams made out of viscoelastic material.

One important type of load that could be applied to a thin-walled structure is an impact load. Mechanical behavior of thin-walled beam under such load was discussed in [10], where six differential equations of motion suitable for some particular dynamic problems were presented. For the case when the center of inertia coincides with the shear center approximate solution for engineering use was obtained.

Generally speaking, the problems of natural and forced oscillations of any structural element are more complex than corresponding static problem and that is why many different engineering techniques were presented by researchers in order to keep up with the growing tendency of high accuracy stress-strain calculations [11–17]. For example, method of additional energy functional, viscoelastic models for parametric nonlinear oscillation problems, response-spectra method for non-proportional damping systems.

The behavior of elastic stress waves in structural elements under dynamic impact, including seismic load, was discussed in [18–20]. In terms of mechanics, the main feature of the problems solved by the authors is the significant influence of forces of inertia applied to flexible bodies. This fact leads to increasing complexity of the governing equations due to their time-dependency and it should be considered whilst solving similar problems with thin-walled bars.

As FEM is discretization method, in most cases its result accuracy depends on the number of elements in specific model. For the vibration problem of thin-walled beam this drawback could be avoided by using dynamic stiffness method presented in [21]. The main idea of this method is to use frequency-dependent shape functions allowing to obtain vibration modes with a higher accuracy than with a classical way of approximation. In this paper it was shown that the curvature of thin-walled beam requires a denser finite element mesh, which makes dynamic stiffness method even more valuable for these kinds of problems.

In [22] mass matrix and stiffness matrix of thin-walled beam in non-shear theory were developed. In this work it was pointed out that FEM matrixes of vibrating rod do not consist of blocks corresponding to different types of deformation, in other words, bending and twisting problems of thin-walled beams are not independent and must be solved simultaneously.

One perspective theory of thin-walled beams of both open and closed cross-section is semi-shear Slivker's theory presented in [23]. The main idea of this theory is to split shear deformation on two groups: torsional and transverse (bending) shear. The last one is neglected just like in Euler-Bernoulli beam theory. Compared to non-shear Vlasov's one this theory has obvious advantage of improved accuracy achieved by shear accounting. Transverse shear neglecting in its turn simplifies governing equations considerably compared to other theories based on Timoshenko's beam model. The only drawback of this theory is more complicated process of sectorial properties calculation, which can be solved by using modern technical computing systems.

The first works devoted to Slivker's semi-shear theory combined with finite element method applied to structural mechanics were [24] and [25]. In this papers with the use of quadratic and linear approximations of twist angle and warping functions several stiffness matrixes of thin-walled beam elements were obtained. Presented numerical and analytical results could be applied for both open and closed cross-sections for rods with different boundary conditions. Moreover, the problem of finding internal forces in thin-walled beam under several types of load were solved in this works.

Fundamental research of all aspects of semi-shear theory in FEM implementation was presented in thesis [26]. The peculiarity of this work, among other things, is that it is supplied with the computer algorithm designed to solve static problems with thin-walled beams.

In work [27] constrained vibrations of thin walled-beams under impulse and short-term loads were discussed. The contact force parameters were determined by means of Hertz's theory, and derived integral equation is solved numerically by Euler's method. With the use of obtained results strength and stiffness of thin-walled beams under dynamic pressure could be determined.

Very important problem concerning thin-walled bars in FEM is connection between finite elements of different types. In [28] it was proposed to model the joints by classical shell elements, whereas the main parts of thin-walled bars are modeled by element with 7 degrees of freedom in each node.

Numerical analysis of spatial thin-walled bar system under torsional loads was carried out in [29, 30]. Specific properties of stiffness matrices of thin-walled rods with opened cross-section were shown and discussed. In addition, the influence of structural steel joints on warping of cross-section was examined.

The equations of motion in TImoshenko's, Vlasov's and Slivker's theories were obtained and compared with each other in work [31]. It was shown that semi-shear theory has an important property of revealing optical dispersion waves in thin-walled beams while the other two theories have only acoustic ones in corresponding spectrum. Optical part of frequencies comes from pure warping vibrations of thin-walled beam which do not show up when shear deformation is neglected. However, mass and stiffness matrixes obtained in this works have a good convergence only if quadratic and hyperbolic approximations are used.

Another work devoted to convergence problem of finite element models of thin-walled beams is [32]. In this article It was shown that linear approximation of both twist angle and warping functions gives a very poor convergence to analytical solutions. One way to improve convergence is to use a higher degree of approximation polynomial. Increasing finite elements mesh density could make numerical results more precise as well. However, this approach has a serious disadvantage of stiffness and mass matrixes enlargement leading to computational expenses. In addition, lack of accuracy problem when calculating stress values still remains.

A detailed review of analytical and numerical calculation methods for static, dynamic and stability problems of thin-walled beams was presented in [33].Due to sufficient computational speed and simplicity of method implementation, FEM was proved to be one of the most perspective techniques for solving different thin-walled beam problems in structural mechanics. In addition, the mentioned article showed that semi-shear theory is the most versatile and mathematically straightforward theory among all the others.

The aim of this research is to construct a finite element model based on mixed variational formulation in order to improve convergence and provide an explicit way to calculate internal forces and stresses in thin-walled bar.

The main objectives of the research are:

1. Writing of the initial mixed functionals of thin-walled bar for static and dynamic problems within semi-shear theory.
2. Derivation of stiffness and mass matrixes with nodal forces vector for single finite element of thin-walled bar.
3. Assembly of global stiffness and mass matrixes with nodal forces vector. Construction of the governing equations of equilibrium and motion.
4. Solving of test static and dynamic problems by mixed FEM with comparison to classical method and analytical solution.

2. Methods

In order to obtain stiffness and mass matrixes based on mixed FEM we need write down Reissner-like functional (see [1]) for thin-walled beam taking into account shear deformation. For the sake of simplicity, we will consider only cross-sections with two symmetry axes. For static problems desired functional can be written as follows:

$$\Phi_{R,stat} = \int_L \left(\frac{M_x^2}{2GI_x} + \frac{B^2}{2EI_\omega} + \frac{M_\omega^2}{2GI_g} - M_x\theta' - B\beta' - M_\omega\left(\theta' - \beta\right) + m_x\theta + b_\omega\beta \right) dL, \tag{1}$$

here I_ω is sectorial moment of inertia, sm^6;

I_x is pure torsional moment of inertia, sm^4;

I_g is constrained torsional moment of inertia, sm^4 (see [26, 31]);

E is elastic modulus, MPa;

G is shear modulus, MPa;

L is beam length, m;

θ is twist angle, rad;

β is warping, rad/m;

M_x is pure torque, N·m;

M_ω is constrained torque, N·m;

B is bimoment, N·m^2;

m_x is distributed torque, N;

b_ω is distributed bimoment, N·m;

Similarly for the case of natural oscillations of thin-walled bar:

$$\Phi_{R,dyn} = \int_L \left(\frac{M_x^2}{2GI_x} + \frac{B^2}{2EI_\omega} + \frac{M_\omega^2}{2GI_g} - M_x\theta' - B\beta' - M_\omega(\theta' - \beta) + \frac{\rho\omega^2 I_r}{2}\theta^2 + \frac{\rho\omega^2 I_\omega}{2}\beta^2 \right) dL, \tag{2}$$

here ρ is material density, kg/m^3;

ω is natural frequency, rad/s;

I_r is polar moment of inertia, sm^4;

In order to obtain stiffness and mass matrixes and force vector, one should define the shape functions for all the unknown values in functionals (1) and (2). From now on, we will use only linear approximation functions:

$$N_1(x) = \frac{x}{L}; \quad N_2(x) = 1 - \frac{x}{L}. \tag{3}$$

For each function $f(x)$: $M_x(x)$, $B(x)$, $M_\omega(x)$, $\theta(x)$, $\beta(x)$ we can write:

$$f(x) = N_1(x) f_1 + N_2(x) f_2, \tag{3a}$$

here f_1, f_2 are the values of function $f(x)$ in the first and second node, respectively.

Substituting all the internal forces and displacements in (1) and (2) by their expressions in terms of shape functions and unknown nodal values (3a), we can use Hamilton's principle to obtain equations of equilibrium and motion. For convenience, we denote (4)–(7):

$$M = \begin{bmatrix} \frac{\rho L I_\omega}{3} & 0 & \frac{\rho L I_\omega}{6} & 0 \\ 0 & \frac{\rho L I_r}{3} & 0 & \frac{\rho L I_r}{6} \\ \frac{\rho L I_\omega}{6} & 0 & \frac{\rho L I_\omega}{3} & 0 \\ 0 & \frac{\rho L I_r}{6} & 0 & \frac{\rho L I_r}{3} \end{bmatrix}, \tag{4}$$

here M is mass matrix block;

$$F = \begin{bmatrix} \frac{L}{3EI_\omega} & 0 & 0 & \frac{L}{6EI_\omega} & 0 & 0 \\ 0 & \frac{L}{3GI_g} & 0 & 0 & \frac{L}{6GI_g} & 0 \\ 0 & 0 & \frac{L}{3GI_x} & 0 & 0 & \frac{L}{6GI_x} \\ \frac{L}{6EI_\omega} & 0 & 0 & \frac{L}{3EI_\omega} & 0 & 0 \\ 0 & \frac{L}{6GI_g} & 0 & 0 & \frac{L}{3GI_g} & 0 \\ 0 & 0 & \frac{L}{6GI_x} & 0 & 0 & \frac{L}{3GI_x} \end{bmatrix}, \tag{5}$$

here F is standard stiffness matrix block;

$$S=\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{L}{3} & -\frac{1}{2} & -\frac{L}{6} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{L}{6} & -\frac{1}{2} & -\frac{L}{3} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \tag{6}$$

here S is mixed stiffness matrix block;

$$P=\begin{bmatrix} \frac{b_\omega L}{2} \\ \frac{m_x L}{2} \\ \frac{b_\omega L}{2} \\ \frac{m_x L}{2} \end{bmatrix}, \tag{7}$$

here P is nodal forces vector block;

Vector blocks of unknown nodal displacements and internal forces are denoted as follows (8)–(9):

$$W=\begin{bmatrix} \beta_1 \\ \theta_1 \\ \beta_2 \\ \theta_2 \end{bmatrix}, \tag{8}$$

here W is nodal displacements vector block;

$$V=\begin{bmatrix} B_1 \\ M_{\omega 1} \\ M_{x1} \\ B_2 \\ M_{\omega 2} \\ M_{x2} \end{bmatrix}, \tag{9}$$

here V is nodal internal forces vector block, index numbers 1 and 2 correspond to the first and second node of finite element, respectively;

With presented designations, functional (1) will look like:

$$\Phi_{R,stat}=\frac{1}{2}V^T FV - V^T SW + W^T P. \tag{9a}$$

Similarly, for expression (2):

$$\Phi_{R,dyn}=\frac{1}{2}V^T FV - V^T SW + \frac{1}{2}\omega^2 W^T MW. \tag{9b}$$

Hamilton's principle for functionals (9a) and (9b) can be written down as follows:

$$\frac{\partial \Phi_R}{\partial V}=0; \frac{\partial \Phi_R}{\partial W}=0. \tag{9c}$$

Conditions (9c) yield the equations of equilibrium and motion of thin-walled bar. In case of static problem, governing system of linear equations will look like:

$$\begin{cases} S^T V = P, \\ -FV + SW = 0. \end{cases} \tag{10}$$

Likewise for natural vibrations of thin-walled bar:

$$\begin{cases} S^T V = \omega^2 M W, \\ -FV + SW = 0. \end{cases} \tag{11}$$

The first obtained system of equations can be written down in a more general way:

$$\begin{bmatrix} S^T & 0 \\ -F & S \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}. \tag{12}$$

Corresponding matrix form of the second system of equations:

$$\left(\begin{bmatrix} S^T & 0 \\ -F & S \end{bmatrix} - \omega^2 \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{13}$$

Thereby, the governing systems of equations for static and dynamic problems of thin-walled beam were obtained.

Now we can introduce the vector of unknown values:

$$U_{element} = \begin{bmatrix} \beta_1 \\ \theta_1 \\ B_1 \\ M_{\omega 1} \\ M_{x1} \\ \beta_2 \\ \theta_2 \\ B_2 \\ M_{\omega 2} \\ M_{x2} \end{bmatrix}. \tag{14}$$

Corresponding stiffness matrix of thin-walled beam element will look like:

$$K_{element} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{L}{3} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{L}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{-L}{3EI_\omega} & 0 & 0 & \frac{1}{2} & 0 & \frac{-L}{6EI_\omega} & 0 & 0 \\ -\frac{L}{3} & -\frac{1}{2} & 0 & \frac{-L}{3GI_g} & 0 & -\frac{L}{6} & \frac{1}{2} & 0 & \frac{-L}{6GI_g} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{-L}{3GI_x} & 0 & \frac{1}{2} & 0 & 0 & \frac{-L}{6GI_x} \\ 0 & 0 & \frac{1}{2} & -\frac{L}{6} & 0 & 0 & 0 & \frac{1}{2} & -\frac{L}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{-L}{6EI_\omega} & 0 & 0 & \frac{1}{2} & 0 & \frac{-L}{3EI_\omega} & 0 & 0 \\ -\frac{L}{6} & -\frac{1}{2} & 0 & \frac{-L}{6GI_g} & 0 & -\frac{L}{3} & \frac{1}{2} & 0 & \frac{-L}{3GI_g} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{-L}{6GI_x} & 0 & \frac{1}{2} & 0 & 0 & \frac{-L}{3GI_x} \end{pmatrix}. \tag{15}$$

Similarly, the mass matrix of the element can be denoted as:

$$M_{element} = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}. \tag{15a}$$

Expressions (12) should be treated as a system of linear equations while (13) is classic eigenvalue problem.

In order to give formulas (12) and (13) more traditional FEM look we can do some mathematical transformations:

$$(S^T F^{-1} S)W = P, \quad (S^T F^{-1} S)W = \omega^2 MW. \tag{16}$$

We can denote:

$$K = (S^T F^{-1} S), \tag{17}$$

here K is shortened version of mixed stiffness matrix.

Therefore (16) will look like:

$$KW = P \tag{18}$$

and:

$$\left(K - \omega^2 M\right)W = 0. \tag{18a}$$

Expressions (18) and (18a) coincide with traditional FEM equations of linear statics and dynamics, but with stiffness matrix based on formula (17).

Obtained matrixes can be used to determine stress-strain state and natural frequencies of thin-walled beam with arbitrary boundary conditions and under any uniformly distributed load.

In order to get exact analytical solutions of static and dynamic problems we have to derive Euler equations using expressions (1) and (2).

In order to apply Hamilton's principle we need to calculate the following values for each function $f(x)$: $M_x(x)$, $B(x)$, $M_\omega(x)$, $\theta(x)$, $\beta(x)$:

$$\frac{\partial \Phi_R}{\partial f\left(x\right)} = 0. \tag{19}$$

For the static case expression (19) gives:

$$\begin{cases} M'_\omega + M'_x = -m_x, \\ B' + M_\omega = -b_\omega, \\ \dfrac{M_\omega}{GI_g} + \beta - \theta' = 0, \\ \dfrac{B}{EI_\omega} - \beta' = 0, \\ \dfrac{M_x}{GI_x} - \theta' = 0. \end{cases} \tag{19}$$

Similarly, for the case of natural vibrations:

$$\begin{cases} M'_\omega + M'_x = -\omega^2 \rho I_r \theta, \\ B' + M_\omega = -\omega^2 \rho I_\omega \beta, \\ \dfrac{M_\omega}{GI_g} + \beta - \theta' = 0, \\ \dfrac{B}{EI_\omega} - \beta' = 0, \\ \dfrac{M_x}{GI_x} - \theta' = 0. \end{cases} \tag{20}$$

Equations (19) and (20) will be used to obtain exact solutions of thin-walled beam problems in order to compare with numerical results.

3. Results and Discussion

Static analysis of thin-walled beam under uniformly distributed torque (Figure 1) will be carried out to examine accuracy of obtained equations (12).

The value of distributed torque m_x is 10 N·m.

Geometrical and mechanical characteristics of the beam are presented in Table 1:

Table 1. Geometrical and mechanical properties of thin-walled beam.

E, Pa	G, Pa	I_x, m^4	I_ω, m^6	A, m^2	L, m	ρ, kg/m^3	I_r, m^4	I_g, m^4
$2.06\cdot10^{11}$	$7.9\cdot10^{10}$	$1.11\cdot10^{-9}$	$1.48\cdot10^{-11}$	$3.51\cdot10^{-4}$	10.0	7850.0	$1.41\cdot10^{-7}$	$1.85\cdot10^{-7}$

Corresponding thin-walled cross-section is depicted on Figure 2.

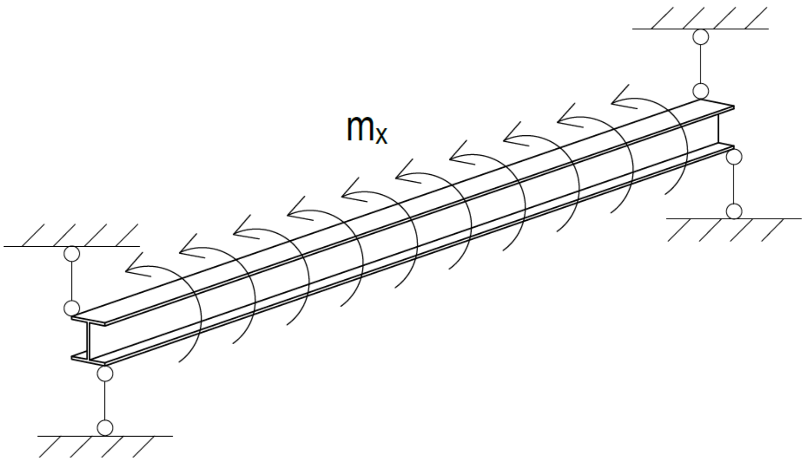

Figure 1. Thin-walled beam under uniformly distributed torque.

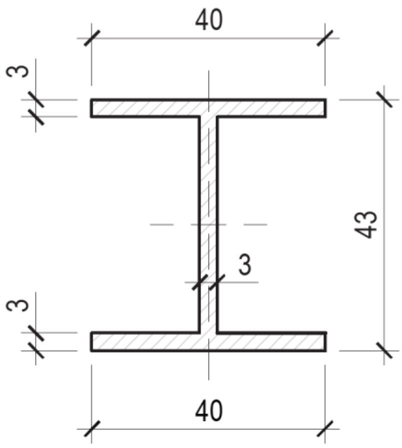

Figure 2. Thin-walled cross-section (dimensions in cm).

The exact analytical solution of equations (19) for the value of bimoment in cross-section of thin-walled beam with boundary conditions on Figure 1 can be written down as follows:

$$B_\omega(x) = \frac{m_x}{GI_x}\left[\frac{ch\left(kx - \frac{kL}{2}\right)}{ch\left(\frac{kL}{2}\right)} - 1\right]EI_\omega, \tag{21}$$

here

$$k = \sqrt{\frac{GI_x I_g}{\left(I_x + I_g\right)EI_\omega}}. \tag{22}$$

For example, we can calculate bimoment value in the middle of beam's span:

$$B_\omega\left(\frac{L}{2}\right) = \frac{m_x EI_\omega}{GI_x}\left[\frac{1}{ch\left(\frac{kL}{2}\right)} - 1\right]. \tag{23}$$

Substituting beam properties into (23) we can calculate bimoment, which equals 0.348 kN·m^2

For the comparison of mixed and classical FEMs to be correct the finite element mesh of thin-walled beam must be chosen in the way that the number of unknown values for both methods approximately equaled to each other. For mixed FEM we have chosen 8 elements which corresponds to 45 unknown values, and for classical FEM – 21 elements (44 unknown values). The first method gives the value of bimoment 0.327 kN·m^2, while the second one – 0.298 kN·m^2.

It appears that mixed FEM has much more accurate results of internal force analysis of thin-walled beam compared to classical method. The relative errors can be calculated as follows:

$$\delta_{mixed} = \frac{B_{L/2}^{mixed} - B_{L/2}^{exact}}{B_{L/2}^{exact}} 100\,\% = \frac{0.327 - 0.348}{0.348} 100\,\% = -6.03\,\%,$$

$$\delta_{class} = \frac{B_{L/2}^{class} - B_{L/2}^{exact}}{B_{L/2}^{exact}} 100\,\% = \frac{0.298 - 0.348}{0.348} 100\,\% = -14.37\,\%.$$

here $B_{L/2}^{mixed}$, $B_{L/2}^{class}$, $B_{L/2}^{exact}$ are bimoments calculated by mixed FEM, classical FEM and analytical formula (23), respectively.

Modal analysis of thin-walled beam depicted on Figure 3 will be carried out to examine convergence properties of (13).

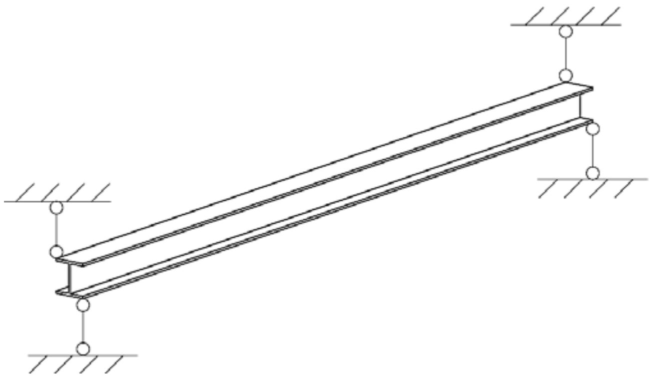

Figure 3. Boundary conditions of thin-walled beam

The exact analytical solution of equations (20) for the acoustic eigenfrequencies of thin-walled beam with boundary conditions on Figure 3 will look like:

$$\omega = \sqrt{\frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\alpha_1}}, \quad (21)$$

here

$$\alpha_1 = \frac{\rho^2 I_r}{GI_g E}, \quad (22)$$

$$\alpha_2 = \frac{\rho\eta^2}{E} + \frac{\rho I_r \eta^2}{GI_g} + \frac{\rho I_r I_g}{\left(I_x + I_g\right) EI_\omega}, \quad (23)$$

$$\alpha_3 = \frac{GI_x I_g \eta^2}{\left(I_x + I_g\right) EI_\omega} + \eta^4, \quad (24)$$

$$\eta = \frac{\pi n}{L}, \quad n \in N. \quad (25)$$

Performing numerical calculations using expression (13) and matrixes (4)–(6) and (8)–(9) followed by determination of the exact solution for the first acoustic frequency by means of (21)–(25) we can draw Table 2:

Table 2. Fundamental frequency by mixed FEM, classical FEM and analytical solution.

Number of finite elements	2	4	8	16	32	64	128
Mixed FEM 1st frequency, rad/s	291.01	135.84	94.81	89.57	88.60	88.47	88.47
Classical FEM 1st frequency, rad/s	636.89	287.10	158.47	109.85	94.26	89.96	88.85
Exact 1st frequency, rad/s				88.48			

Graphical visualization of the numerical results is presented on Figure 4.

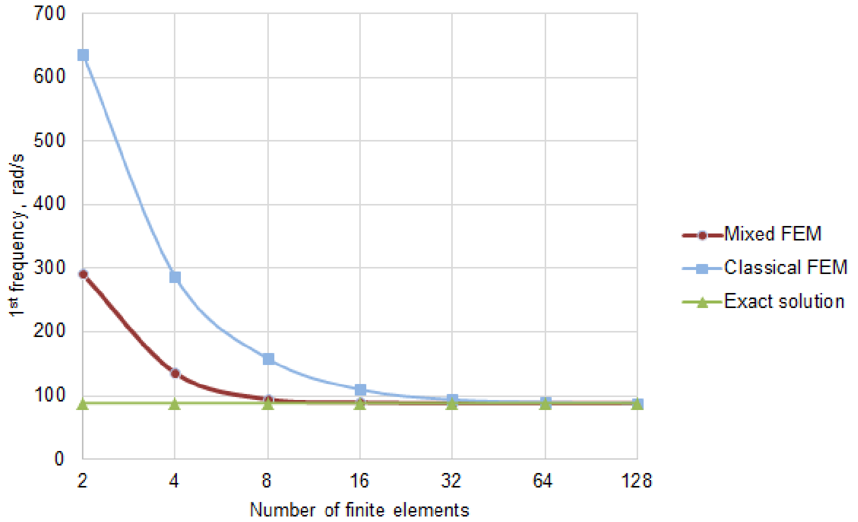

Figure 4. Convergence of fundamental frequency by mixed and classical FEM to analytical solution.

Table 2 and Figure 4 show that mixed FEM gives much more accurate value of the first eigenfrequency of thin-walled beam compared to classical FEM. For example, 16 finite elements produce only 1.23 % error:

$$\delta = \frac{\omega_1^{mixed} - \omega_1^{exact}}{\omega_1^{exact}} 100\ \% = \frac{89.57 - 88.48}{88.48} 100\ \% = 1.23\ \%.$$

While classical FEM with 16 elements gives 24.15 %:

$$\delta = \frac{\omega_1^{class} - \omega_1^{exact}}{\omega_1^{exact}} 100\ \% = \frac{109.85 - 88.48}{88.48} 100\ \% = 24.15\ \%,$$

here ω_1^{mixed}, ω_1^{class}, ω_1^{exact} are fundamental frequencies calculated by mixed FEM, classical FEM and analytical formula (21), respectively.

Comparing these values to the results of work [31], where classical FEM with linear approximation and 32 finite elements has given an error of 23.3 % for similar eigenfrequency problem of thin-walled beam, we can see that developed mixed FEM model produces a much more precise outcome and has a better convergence.

The other important property of mixed FEM applied to dynamic problems of thin-walled beam is that the vibration modes include not only warping and twist angle functions, but internal torques and bimoment as well. The first mode shape of the examined beam with 16 finite elements is presented on Figures 5–9.

Besides the advantages of high accuracy of calculations and explicit output of internal forces, there is also one drawback concerning dynamic analysis of thin-walled beam using mixed FEM. If not only the fundamental mode shape is considered, the effect of “spurious” high frequencies in thin-walled beam spectrum occurs. In other words, the frequencies are not arranged in the ascending order if mixed FEM is used. For the examined beam with 16 finite elements, it appears that the third and seventh mode shapes are presented by high-frequency vibrations (Figures 10, 11).

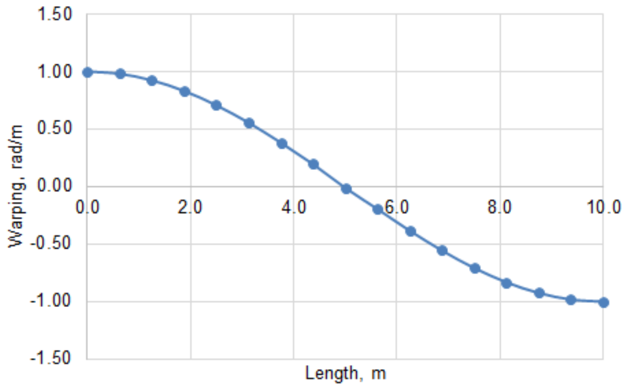

Figure 5. The first warping mode of thin-walled beam.

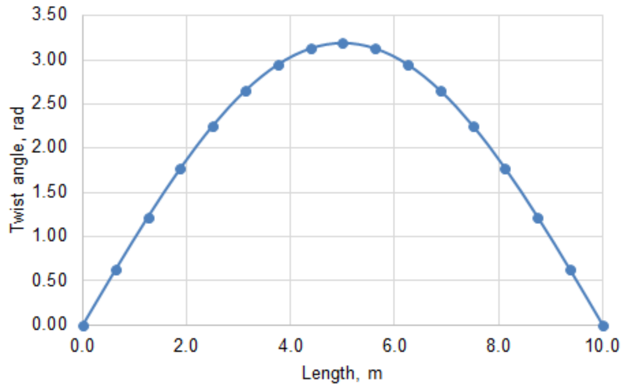

Figure 6. The first twist angle mode of thin-walled beam.

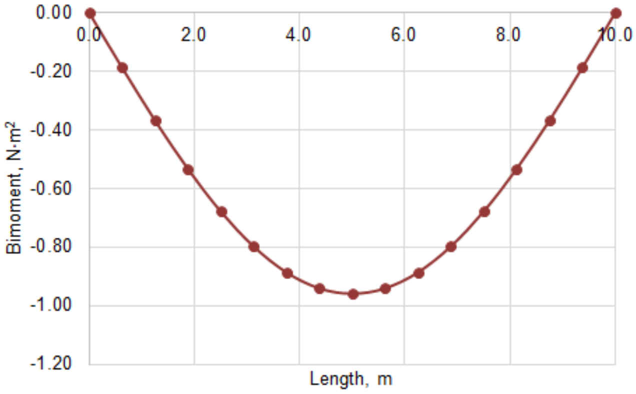

Figure 7. The first bimoment mode of thin-walled beam.

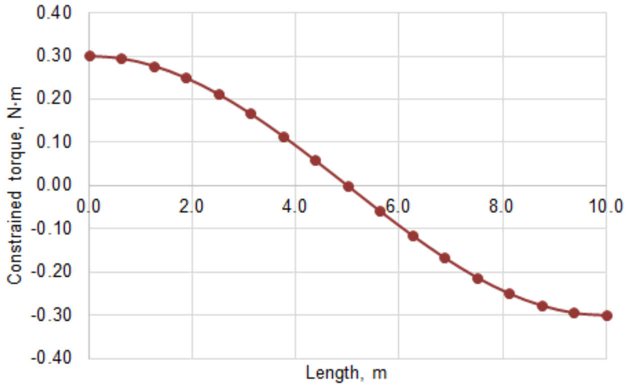

Figure 8. The first constrained torque mode of thin-walled beam.

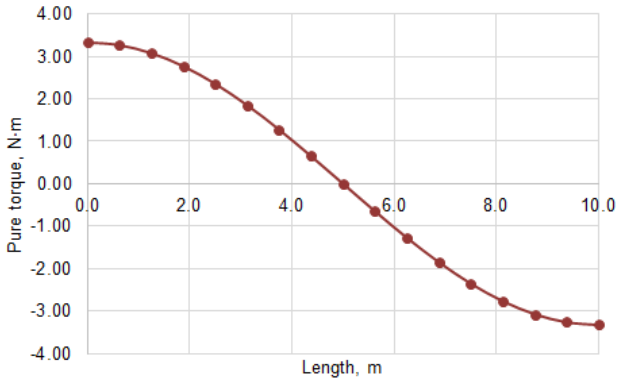

Figure 9. The first pure torque mode of thin-walled beam.

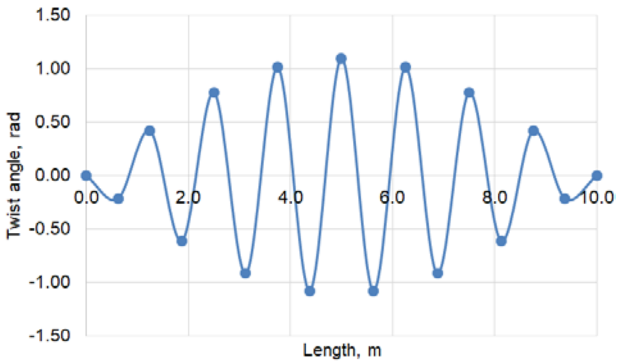

Figure 10. The 3rd twist angle mode of thin-walled beam.

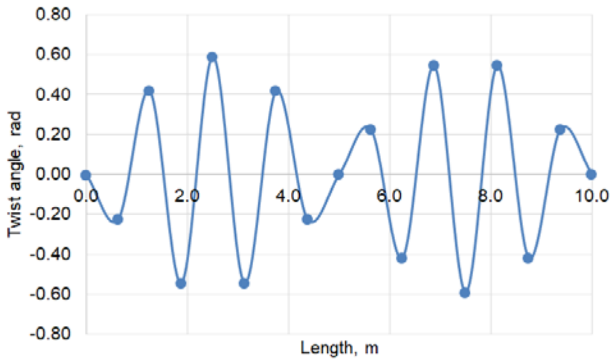

Figure 11. The 7th twist angle mode of thin-walled beam.

The presented effect can bring some difficulties in modal analysis of thin-walled beam as the “spurious” frequencies distort the spectrum, making it necessary to filter them out.

The observed problem could be easily solved in practical application by means of simple visual analysis of oscillation modes, since the “spurious” frequency can always be detected by many sign alterations of the corresponding mode.

It should be noted that the proposed model of thin-walled rod based on mixed formulation could be generalized for more complex spatial bar systems. Although, the matrix for transformation from local coordinate systems to the global one should in some way take into account the dependence between relative angle positions of rods and warping of cross-sections. At the present time, the problem of bimoment and warping “rotation” in nodal parts of thin-walled bar systems stays unresolved. This is very perspective scientific challenge and when some progress in this direction is achieved the results of the article could be applied to flat and spatial thin-walled bar systems.

4. Conclusions

Mixed variational formulation of static and dynamic problems of thin-walled bars has many advantages compared to classical approach. The main ones can be summarized as follows:

1. Mixed FEM gives more accurate results of stress analysis of thin-walled bar under static load. For example, the value of bimoment in the middle of the bar under uniform torque matches the analytical solution

with relative error of only 6.03 %, while classical FEM gives the tolerance of 14.37 %, whereas the number of unknown values is 45 and 44, respectively.

2. The internal forces evaluation proceeds simultaneously with the main operation of solving system of linear equations, which obviates the need to perform additional calculations.

3. Natural mode shapes and eigenfrequencies of thin-walled bar are determined with a higher accuracy, which allows to reduce the number of finite elements in the model. For example, the value of fundamental frequency of the examined bar with 16 finite elements matches the analytical solution with relative error of only 1.23 %, while classical FEM gives the tolerance of 24.15 %.

However, mixed FEM has a drawback of revealing “spurious” high-frequency vibration modes distorting spectrum of thin-walled beams.

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Contacts:

Vladimir Lalin, +7(921)3199878; vllalin@yandex.ru

Vladimir Rybakov, +7(964)3312915; fishermanoff@mail.ru

Sergey Ivanov, +7(904)5567654; serzikserzik@gmail.com

Artur Azarov, +7(905)2705646; alexio009@mail.ru

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Смешанный метод конечных элементов в полусдвиговой теории тонкостенных стержней В.И. Сливкера

В.В. Лалин, В.А. Рыбаков*, С.С. Иванов, А.А. Азаров
Санкт-Петербургский политехнический университет Петра Великого, Санкт-Петербург, Россия
** E-mail: fishermanoff@mail.ru*

Ключевые слова: смешанный метод конечных элементов, тонкостенный стержень, полусдвиговая теория Сливкера, функционал Рейсснера

Аннотация. Представлена смешанная вариационная постановка задач статики и динамики тонкостенных стержней. Получены матрицы жесткости и матрицы масс конечного элемента на основе выражения, аналогичного функционалу Рейсснера. С помощью полусдвиговой теории Сливкера произведен учет деформации сдвига в поперечном сечении тонкостенного стержня. Из предложенного смешанного функционала получены соответствующие уравнения Эйлера. В качестве аппроксимации искомых функций перемещений и внутренних усилий рассмотрены линейные полиномы Эрмита. Для некоторых частных статических и динамических задач тонкостенных стержней представлено точное аналитическое решение, основанное на уравнениях Эйлера. Продемонстрирован эффект появления “лишних” частот в спектре тонкостенного стержня при использовании смешанной конечно-элементной постановки. Произведено численное сравнение результатов расчета смешанным и классическим методами конечных элементов.

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Контактные данные:

Владимир Владимирович Лалин, +7(921)3199878; эл. почта: vllalin@yandex.ru

Владимир Александрович Рыбаков, +7(964)3312915; эл. почта: fishermanoff@mail.ru

Сергей Сергеевич Иванов, +7(904)5567654; эл. почта: serzikserzik@gmail.com

Артур Александрович Азаров, +7(905)2705646; эл. почта: alexio009@mail.ru