Magazine of Civil Engineering

ISSN 2712-8172

journal homepage: http://engstroy.spbstu.ru/

Research article

UDC 624.074.3:539.3

DOI: 10.34910/MCE.116.5

Stress condition of orthotropic vault structure with cylindrical anisotropy

***A-Kh.B. Kaldar-ool*[1] ✉, *E.K. Opbul*[2]**

[1] *Tuvan State University, Kyzyl. Republic of Tuva, Russia, oorzhaka-h@mail.ru, 0000-0002-5304-3747*

[2] *St. Petersburg State University of Architecture and Civil Engineering, St. Petersburg, Russia*

✉ *oorzhaka-h@mail.ru*

Keywords: barrel vaults, brickwork, anisotropy, elasticity modulus, elasticity constants, stress condition

Abstract. This work considers analytical calculation of brickwork barrel vault, material structure of which has a pronounced variability of elastic constants. In normative documents, brickwork is considered as a complex two-component building material with elastoplastic properties. However, there are no clear recommendations that consider the variability of the elastic properties of brickwork. This article considers influence of anisotropic properties of a brickwork three-centered flat-arched vault on its stress condition on the basis of the elasticity theory. The calculation of a flat-arched vault is based on the classic theory of bending a curved curvilinearly-anisotropic beam in view of the properties of brickwork materials with cylindrical anisotropy. We cite a mathematical solution of a differential equation of fourth order in partial derivative with two variables for an anisotropic orthotropic body in polar coordinates for creation of mathematical models describing changes in the vault material elasticity modulus. Based on the solution to the curved orthotropic body anisotropy problem, we obtained correlations between elastic constants in the main anisotropy directions.

Citation: Kaldar-ool, A-Kh.B., Opbul, E.K. Stress condition of orthotropic vault structure with cylindrical anisotropy. Magazine of Civil Engineering. 2022. 116(8). Article no. 11605. DOI: 10.34910/MCE.116.5

1. Introduction

Study object is brickwork barrel vault of complex curvilinear outline. Brickwork is an orthotropic two-component material with elastoplastic properties. In vaulted structures, depending on the direction, in a three-dimensional coordinate system, brickwork has different elastic properties. In the case of two-dimensional problem, variability manifests itself in tangential (circumferential) and radial directions.

It is well known that elastic constants include the values of the modulus of elasticity and Poisson's ratio of the masonry. Ignoring the elastic properties variability factor in calculations, for example, at the design stage, can lead to serious errors or accidents during operation. Moreover, in brickwork vaulted ceilings, anisotropy is observed not only in the material itself, but also in structural layout. In view of more frequent progressive collapses, theoretical studies based on calculation of the strength and stress state of stone vaulted structures, taking into account the properties of anisotropy, are extremely critical.

According to Professor S.G. Lekhnitsky's theory [1], brickwork barrel vaults [2] can be viewed as an orthotropic body with cylindrical anisotropy [3] for which laws of mechanics of anisotropic bodies are valid.

In the classical practice of constructing buildings and structures, including brickwork ceilings, load-bearing vaults of different geometric shapes and layouts were used. Methods of their calculation were mainly based on the laws of elasticity theory, considering the vault as a hinged-supported elastic body. Approximate methods of limiting equilibrium, proposed by Maurice Levy, and a graphical method for determining the pressure curve, based on experiments on the destruction of vaults, are used.

A particularly noteworthy study [4] examines the technical condition of historical buildings, where strength characteristics of brickwork vaulted structures is assessed on the basis of numerical modeling. The study [5] considers flat circular double-hinged vaults, statically loaded with a uniformly distributed load in the form of pressure. The authors managed to analytically solve the geometrically nonlinear deformability and stability of the vaults. Behavior of Prussian brick-concrete vaults based on numerical model built in the Abaqus software packageis shown in a scientific paper [6]. Finite element analysis of the behavior of a stonework single-span arched bridge is presented in a scientific study [7]. Laboratory and numerical analysis of the destruction of stone arches on models with and without reinforcement can be found in a scientific study [8]. In the original paper [9], the behavior of a segment of an arch from a dry joint of masonry subjected to a lateral load was investigated using the finite element method and rigid blocks based on a micromodeling strategy, a nonlinear analysis was also presented taking into account various combinations of displacement of the supports (vertical, horizontal and oblique). The article [10] presents the original results of experimental studies to reduce the seismic vulnerability of stone vaults using a composite reinforced mortar. The authors of work [11] argue that displacements, observed in many historical masonry structures are concentrated at the joints of stonework and can be significant before collapse becomes a global problem. Therefore the stability modeling using discrete element modeling is particularly relevant. In order to reuse the building complex of Monte Pio port and the upper warehouses of the Palazzo Monte di Pieta, in work [12], numerical studies of two-level stone vaults were carried out and the technical condition was checked for the stability of building structures in accordance with the current Italian building codes. In scientific work [13], based on analytical formulation of the problem, the angle of friction is investigated as a geometric constraint of brickwork sample in order to find a possible range of minimum values of the thickness of round and elliptical vaults from masonry under static loads based on the limit state analysis theorem. Numerical studies of the stability of masonry vaults are presented in works [14, 15] including a unified formulation of historical stone structures modeled as 2D assemblies of rigid blocks interacting on frictional contact surfaces without stress. Experimental evaluation and development of numerical and analytical modeling of ancient masonry vaults and vaults, reinforced with composite systems, are given in work [16]. The study [17] examines the effect of stereotomy on the value of the minimum thickness of a semicircular brickwork vault: a material with low tensile strength. In the scientific work [18], on the basis of historical, experimental and numerical analysis and the results of field studies using our own software LiABlock_3D, a spatial rigid block model of the entire structural unit was built. The work [19] proposes combined precast and cast-in-sit construction of metro station in Shanghai in the form of a large-span underground vault of 56 elements of ribbed arched segments to be assembled as triple-hinged arches. In the construction of large diameter tunnels, a new mechanized technology for the construction of precast arches has been proposed [20]. For a preliminary assessment of damage in the form of cracks, for example, in the process of mechanized construction, numerical and laboratory experiments were carried out on an arch made of volcanic rock, i.e. tuff; the research results are given in work [21]. Work [22] represents experimental studies of stonework cylindrical vaults made of thin brick tiles; the vaulted structure is considered as a permanent formwork with subsequent reinforcement based on reinforced concrete.

The purpose of the study is to create a calculation a system based on the well-known theory of bending of a curved anisotropic beams, given in work [1].

To achieve this goal, the following **problems** are solved:

1. Determining the elasticity moduli of bricks and mortar.

2. Determining the elasticity modulus of brickwork and the anisotropic index depending on the main anisotropy directions.

3. Determining the elasticity parameters of brickwork that allow to obtain elastic constants of the vault material.

4. Analytical calculation for evaluating the stress condition of a barrel vault in the form of a curved orthotropic beam with a cylindrical anisotropy.

2. Methods

According to regulatory document [23], the elastic modulus of soft-mud bricks E_0^{brick} is determined on the basis of deformation of cubes or prisms cut from bricks using the formula:

$$E_0^{brick} = 200 \div 1200 \cdot R_{brick}, \quad (1)$$

where R_{brick} is brick compressive strength [2, 3].

In a residential buildings design manual [25], the elastic modulus of mortar E_0^{mortar} in a compressed bed joint is determined using the formula:

$$E_0^{mortar} = \frac{t_{mortar}}{\lambda_{mortar}}, \tag{2}$$

where t_{mortar} is the joint thickness; λ_{mortar} is compression compliance of a manually laid horizontal mortar joint under short-term loads determined according to the formula:

$$\lambda_{mortar} = 1.5 \cdot 10^{-3} \cdot R_{mortar}^{-\frac{2}{3}} \cdot t_{mortar},$$

R_{mortar} is the mortar strength limit [2, 3].

Then within the framework of the task at hand, we use the phenomenological method of rheology in order to find out the brickwork elasticity modulus on the main axes of anisotropy.

According to, we can reconstruct the real pattern of material behavior under load using more complicated schemes combining elastic and viscous elements. If we take elastic (bricks) and viscous (mortar) materials and connect them in parallel (at the vault head) and in series (at the vault abutments), we obtain rheological models: Kelvin and Maxwell bodies in the following form [3].

For parallel connection of elements:

$$E_r = E_0^{brick} + E_0^{mortar}. \tag{3}$$

For series connection of elements:

$$E_t = \frac{E_0^{brick} \cdot E_0^{mortar}}{E_0^{brick} + E_0^{mortar}}, \tag{4}$$

where E_r, E_t are constant elasticity modulus of the brickwork anisotropy, respectively, in the radial and tangential (circular) directions.

A partial fourth-order differential equation in polar coordinates for anisotropic body has the following form[1]:

$$\begin{gathered}\frac{1}{E_t} \cdot \frac{\partial^4 F}{\partial r^4} + \left(\frac{1}{G_{rt}} - 2\frac{\mu_{rt}}{E_r}\right) \cdot \frac{1}{r^2} \frac{\partial^4 F}{\partial r^2 \partial \theta^2} + \frac{1}{E_r} \cdot \frac{1}{r^4} \frac{\partial^4 F}{\partial \theta^4} + \\ + \frac{2}{E_t} \cdot \frac{1}{r} \frac{\partial^3 F}{\partial r^3} - \left(\frac{1}{G_{rt}} - 2\frac{\mu_{rt}}{E_r}\right) \frac{1}{r^3} \frac{\partial^2 F}{\partial r \partial \theta^2} - \frac{1}{E_r} \cdot \frac{1}{r^2} \frac{\partial^2 F}{\partial r^2} + \\ + \left(2\frac{1-\mu_{rt}}{E_r} + \frac{1}{G_{rt}}\right) \frac{1}{r^4} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{E_r} \cdot \frac{1}{r^3} \frac{\partial F}{\partial r} = 0,\end{gathered} \tag{5}$$

where μ_{rt}, G_{rt} are Poisson's ratio and elastic modulus, respectively.

When solving equation (5) in plane elastic problem for a circular plate with cylindrical anisotropy, the stress function was taken as a sum of polynomials proposed by E.K. Ashkenazi:

$$F = \sum_{i-1}^{n} x^k \cdot f_k(y), \tag{6}$$

where $f_k(y)$ is an unknown function satisfying differential equation (5).

Solving equation (5) with the substitution of the corresponding derivatives of the stress function (6) and rearrangement result in second-order algebraic equation (7), the roots of which are respectively equal [26]:

$$B^2 - \frac{2}{3}\left(5 + k^2\right)B - \frac{5}{3}k^4 + \frac{14}{3}k^2 + 1 = 0. \quad (7)$$

Professor V.N. Glukhikh states in his studies [26] that cylindrically anisotropic materials differ by elasticity parameters $\left(B_{(1)},\ B_{(2)}\right)$ and can be divided into 2 groups:

- for the first group, elasticity parameter is characterized by three extremes as axes turn from the radial direction to the tangential one, i.e. from 0° to 90°:

$$B_{(1)} = 3 - k^2; \quad (8)$$

- the second group with two extremes:

$$B_{(2)} = \frac{1 + 5k^2}{3}. \quad (9)$$

The studying the plane stress condition of an anisotropic material often encounters the following problem: the elastic constants are known for a certain coordinate system $x,\ y$ and it is required to find the elastic constants for a new coordinate system $x',\ y'.$ For an orthotropic body, it is inconvenient to use the main coordinate system and recalculate the elastic constants [1].

The known formulas as per [26] in view of elasticity parameter $B_{(1)}$ of anisotropic bodies for elasticity modulus $E_{x'}$, shear modulus $G_{x'y'}$, Poisson's ratio $\mu_{x'y'}$, will have a simpler form, which is of prime importance for engineering calculations:

$$\frac{1}{E_{x'}} = \frac{\cos^4\theta}{E_r} + \frac{\sin^4\theta}{E_t} + \frac{B_{(1)}}{E_t}\cdot\cos^2\theta\cdot\sin^2\theta; \quad (10)$$

$$\frac{1}{G_{x'y'}} = \frac{8\left(k^2 - 1\right)}{E_t}\cdot\sin^2\theta\cdot\cos^2\theta + \frac{1}{G_{rt}}; \quad (11)$$

$$\mu_{x'y'} = -E_{x'}\left[\frac{2\left(k^2 - 1\right)}{E_r}\cdot\sin^2\theta\cdot\cos^2\theta + \frac{\mu_{rt}}{E_t}\right], \quad (12)$$

where $G_{rt} = \dfrac{E_t}{3 - k^2 + 2\cdot\mu_{rt}}$ is shear modulus; $G_{x'y'}$ and $\mu_{x'y'}$ respectively, the shear modulus and Poisson's ratio in a new coordinate system.

Similarly, studies [26] recommend calculation formulas for anisotropic materials belonging to the 2nd group $B_{(2)}$:

$$\frac{1}{E_{x'}} = \frac{\cos^4\theta}{E_r} + \frac{\sin^4\theta}{E_t} + B_{(2)}\cdot\frac{1}{E_t}\cdot\sin^2\theta\cdot\cos^2\theta; \quad (13)$$

$$\frac{1}{G_{x'y'}} = \frac{8\cdot\left(1 - k^2\right)}{3\cdot E_t}\cdot\sin^2\theta\cdot\cos^2\theta + \frac{1}{G_{rt}}; \quad (14)$$

$$\mu_{x'y'} = -E_{x'}\left[\frac{2\cdot\left(1 - k^2\right)}{3\cdot E_t}\cdot\sin^2\theta\cdot\cos^2\theta - \frac{\mu_{rt}}{E_t}\right], \quad (15)$$

where $G_{rt} = \dfrac{3 \cdot E_t}{1 + 5 \cdot k^2 + 6 \cdot \mu_{rt}}$ is shear modulus; $G_{x'y'}$ and $\mu_{x'y'}$ respectively, the shear modulus and Poisson's ratio in a new coordinate system.

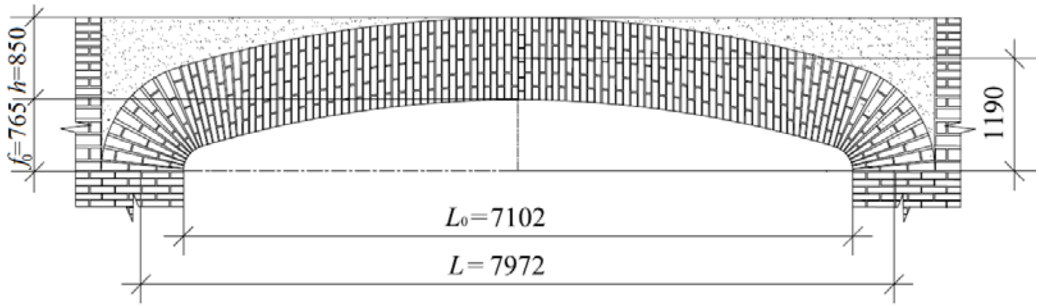

Figure 1. General view of the barrel vault.

A body with cylindrical anisotropy can be formed artificially by constructing it from homogeneous (rectilinear-anisotropic) elements that have the same elastic properties. If we consider that vault consists of a large number of homogeneous anisotropic elements with the same elastic properties, then the structure as a whole will have the property of a body with cylindrical anisotropy. Equivalent axial directions of elements in the vault will be radial directions [1].

For an orthotropic beam with a cylindrical anisotropy in application to a barrelvault we adopt a design diagram presented in Fig. 2.

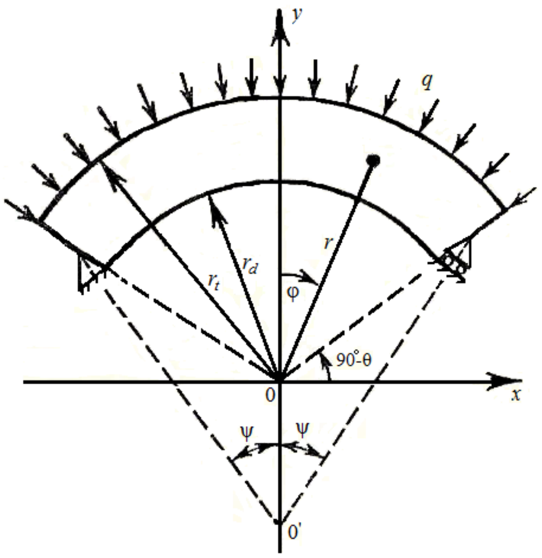

Figure 2. Design diagram of a vault in the form of a curved orthotropic beam with a cylindrical anisotropy: 0 – center of anisotropy coinciding with the center of curvature, 0′ – determining the position of abutment surfaces.

For determining the main stresses in the crest part a barrel vault under the impact of dead (q) and temporary additional loads (F) according to the theory of S.G. Lekhnitsky [1], we use stress function in the following form:

$$F(r) = f_0(r) + f_1(r) \cdot \cos(\theta) + f_1^*(r) \cdot \sin(\theta), \tag{16}$$

where the first addend has the following form:

$$f_0(r) = A + B \cdot r^2 + C \cdot r^{1+k} + D \cdot r^{1-k}, \tag{17}$$

$$k = \sqrt{\frac{E_t}{E_r}}. \tag{18}$$

The second addend for an orthotropic beam is determined by the following formula:

$$f_1(r)\cdot\cos(\theta)=\left(A\cdot r^{1+\beta_1}+B\cdot r^{1-\beta_1}+C\cdot r+D\cdot r\cdot\ln r\right)\cdot\cos\theta+ \\ +\left(A'\cdot r^{1+\beta_1}+B'\cdot r^{1-\beta_1}+C'\cdot r+D'\cdot r\cdot\ln r\right)\cdot\sin\theta, \quad (19)$$

where A, B, C, D are arbitrary complex constant determined from boundary conditions and free-end conditions; A', B', C', D' are conjugate values.

Stresses σ_r, σ_θ, $\tau_{r\theta}$ are expressed through the stress function:

$$\begin{cases}\sigma_r=\dfrac{1}{r}\cdot\dfrac{\partial F(r)}{\partial r}+\dfrac{1}{r^2}\cdot\dfrac{\partial^2F(r)}{\partial\theta^2}\\ \sigma_\theta=\dfrac{\partial^2F(r)}{\partial r^2}\\ \tau_{r\theta}=-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\cdot\dfrac{\partial F(r)}{\partial\theta}\right)\end{cases} \quad (20)$$

Boundary conditions:

$$\begin{cases}\text{when } \ r=r_d\sigma_r=0,\ \tau_{r\theta}=0;\\ \text{when } \ r=r_t\sigma_r=-(q+F)\cos\varphi\tau_{r\theta}=0.\end{cases} \quad (19)$$

Then the condition is true:

$$\left.\begin{aligned}&\int_{r_d}^{r_t}\sigma_\theta\partial r=-\frac{R_\theta}{h}\\ &\int_{r_d}^{r_t}\sigma_\theta r\partial r=0\\ &\int_{r_d}^{r_t}\tau_{r\theta}\partial r=\pm\frac{R_r}{h}\end{aligned}\right\}. \quad (21)$$

Arbitrary constants included in expression (19) are determined on the basis of boundary conditions (21) and (22).

Normal stresses in the main directions are determined according to [1] by formulas (23), (24) using the Mathcad software system:

$$\left.\begin{aligned}&\sigma_r=\frac{q}{b}\cdot\left[P+Q\cdot\left(\frac{r}{r_t}\right)^{k-1}+R\cdot\left(\frac{r_t}{r}\right)^{k+1}\right]+\\ &\quad+\frac{q}{r_t\cdot b\cdot g_1}\cdot\frac{r_t}{r}\cdot\left[\left(\frac{r}{r_t}\right)^{\beta_1}+c^{\beta_1}\cdot\left(\frac{r_t}{r}\right)^{\beta_1}-\left(1+c^{\beta_1}\right)\right]\cdot\frac{\cos(\phi-\psi)}{\cos\psi}\cdot\cos\theta\\ &\sigma_\theta=\frac{q}{b}\cdot\left[P+Q\cdot k\cdot\left(\frac{r}{r_t}\right)^{k-1}+R\cdot k\cdot\left(\frac{r_t}{r}\right)^{k+1}\right]+\\ &\quad+\frac{q}{r_t\cdot b\cdot g_1}\cdot\frac{r_t}{r}\cdot\left[(1+\beta_1)\cdot\left(\frac{r}{r_t}\right)^{\beta_1}+(1+\beta_1)\cdot c^{\beta_1}\cdot\left(\frac{r_t}{r}\right)^{\beta_1}-\left(1+c^{\beta_1}\right)\right]\cdot\frac{\cos(\phi-\psi)}{\cos\psi}\cdot\cos\theta\\ &\tau_{r\theta}=\frac{q}{r_t\cdot b\cdot g_1}\cdot\frac{r_t}{r}\cdot\left[\left(\frac{r}{r_t}\right)^{\beta_1}+c^{\beta_1}\cdot\left(\frac{r_t}{r}\right)^{\beta_1}-\left(1+c^{\beta_1}\right)\right]\cdot\frac{\cos(\phi-\psi)}{\cos\psi}\cdot\cos\theta\end{aligned}\right\} \quad (23)$$

where

$$\left.\begin{aligned} P &= \frac{1}{2\cdot(k^2-1)\cdot(1-c^{2k})\cdot g}\cdot[2\cdot k\cdot(k-1)\cdot(1-c^{k+1})+2\cdot k\cdot(k+1)\cdot c^{k+1}\cdot(1-c^{k-1})- \\ &\quad -(k^2-1)\cdot(1+c)\cdot(1-c^{2k})\cdot m], \\ Q &= \frac{1}{2\cdot(k-1)\cdot(1-c^{2k})\cdot g}\cdot[-(k-1)\cdot(1-c^2)-2\cdot k\cdot c^2\cdot(1-c^{k-1})+(k-1)\cdot(1+c)\cdot \\ &\quad \cdot(1-c^{k+1})\cdot m], \\ R &= \frac{1}{2\cdot(k+1)\cdot(1-c^{2k})\cdot g}\cdot[(k+1)\cdot c^{2k}\cdot(1-c^2)-2\cdot k\cdot c^{k+1}\cdot(1-c^{k+1})-(k+1)\cdot \\ &\quad \cdot(1+c)\cdot c^{2k}\cdot(1-c^{1-k})\cdot m] \end{aligned}\right\} \quad (24)$$

$c=\dfrac{r_d}{r_t}$; $m=\dfrac{\sin\varphi\cdot\sin(\varphi-\psi)}{\cos\psi}$; $g=\dfrac{(1-c^2)}{2}-\frac{k}{k+1}\cdot\frac{(1-c^{k+1})^2}{1-c^2}+\dfrac{k\cdot c^2}{k-1}\cdot\frac{(1-c^{k-1})^2}{1-c^{2k}}$;

$g_1=\dfrac{2}{\beta_1}\cdot(1-c^{\beta_1})+(1+c^{\beta_1})\cdot\ln c$; $b=1$ – a single element width.

It is stated in study [1] that coefficient β_1 included in formula (23):

$$\beta_1=\sqrt{1+\frac{E_t}{E_r}\cdot(1-2\cdot\mu_{rt})+\frac{E_t}{G_{rt}}} \quad (25)$$

3. Results and Discussion

Initial data for practical calculation:

Elastic modulus of a brick of grade Mbrick = 102 and R_{brick} = 10.03 MPa according to [2], obtained by the formula (1) is E_0^{brick} = 12036 MPa.

Elastic modulus of the solution of grade Mmortar = 22 and R_{mortar} = 2.14 MPa, according to the formula (2) is E_0^{mortar} = 2140 MPa.

Radial and tangential moduli of elasticity of brickwork according to formulas (3, 4), respectively E_r = 14176 MPa and E_t = 1816 MPa.

Poisson's ratio according to K.P. Yakovlev [27] – μ_{rt} = 0.15

For a practical calculation example, Table 1 shows the collection of loads.

Table 1. Collection of loads per 1 running meter barrel vault in section at the crest part.

№	Name	Regulatory load, kN/m	Coefficient reliability γ_f	Estimated load, kN/m
1	Curb self weight $q_{c.s.w.}$ (γ = 18 kN/m^3) 18·h in the crest part	15.3	1.1	16.83
2	Backfill weight $q_{b.w.}$ (γ = 18 kN/m^3)	Variable	1.3	–
3	Ground floor weight $q_{g.f.}$ t=55 m: cement-sand mortar t=28.4 mm (γ=18 kN/m^3) marble chips t =15 mm (γ = 16 kN/m^3)	0.511 0.24	1.3	0.976
4	live load [8.2, table.8.3, p. 3]	2	1.2	2.4
5	Experimental live load	–	–	10.0
	Total:			30.206

The values of the vault loads in the crest part per running meter at an angle φ = 0° (see Fig. 3) and according to Table 1 are q = 30.206 kN/m.

If the supporting surfaces are assumed to be horizontal (Fig. 3), then we have φ = 0,ψ = 0, $\theta = \pi/2$, $\cos(\varphi - \psi) = 0$.

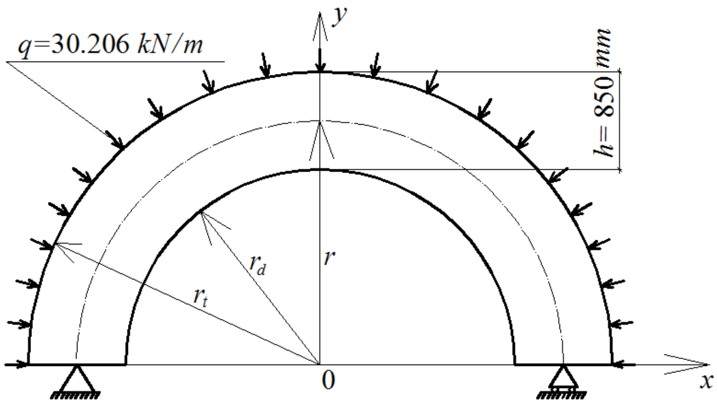

Figure 3. Calculation scheme of the vault.

Bending of curvilinearly anisotropic beam: the radius axis of slope part of vault r = 12.16 m; vault inner radius r_d = 11.73 m, outer radius r_t = 12.58 m.

1. Using known formulas (10, 13), we find out the elastic constants of brick and mortar, including for brickwork.

The values of theoretical elastic moduli of brick and mortarare presented in Tables 2–5.

Table 2. Theoretical elastic modulus of brick at $B_{(1)} = 3 - k^2$ ***according to the formula (1), (MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
k^2=0.5	24072	20190	14810	12036	11330	11700	12036

Table 3. Theoretical elastic modulus of brick at $B_{(2)} = \frac{1+5k^2}{3}$ ***according to the formula (1), (MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
k^2=0.5	24072	23480	21400	18050	14810	12730	12036

Table 4. Theoretical elastic modulus of mortar at $B_{(1)} = 3 - k^2$ ***according to formula (2), (MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
k^2=0.5	4280	3591	2634	2140	2014	2080	2140

Table 5. Theoretical elastic modulus of mortar at $B_{(2)} = \frac{1+5k^2}{3}$ ***according to formula (2), (MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
k^2=0.5	4280	4174	3804	3210	2634	2263	2140

The values of theoretical brickwork elasticity moduli are presented in Tables 6, 7.

Table 6. Theoretical brickwork elasticity modulus at $B_{(1)} = 3 - k^2$ ***(E₀=12036 MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
According to formulas (10, 13) k^2=0.128	14176	6146	2698	1816	1638	1729	1816

Table 7. Theoretical brickwork elasticity modulus at $B_{(2)} = \frac{1+5k^2}{3}$ ***(MPa).***

θ°	0°	15°	30°	45°	60°	75°	90°
According to formulas (10, 13) k^2=0.128	14176	12092	7660	4337	2698	2006	1816

2. The coefficient is determined β_1.

According to [1], for parameter $B_{(1)}$ we have:

$$\frac{E_t}{G_{rt}} - 2 \cdot \mu_{rt} \frac{E_t}{E_r} = B_{(1)}. \tag{26}$$

With regard for expression (18), (26) and (8) obtained in study [26], from formula (25) we get a parameter equal to:

$$\beta_1 = \sqrt{1 + k^2 + 3 - k^2} = 2.$$

At β_1 = 2 stress distribution is analogous to that in an isotropic beam. Studies [26] confirmed relationship (8) mathematically.

If elastic constants meet the condition:

$$\frac{E_t}{E_r}\left(1 - 2 \cdot \mu_{rt}\right) + \frac{E_t}{G_{rt}} = 3, \tag{27}$$

then β_1 = 2 and stress distribution will be exactly the same as in an isotropic beam.

If we transform the radical expression (23) for β_1, we obtain from solution [1] the same root from formula (8), i.e. $\frac{E_t}{E_r} - 2 \cdot \mu_{rt} \cdot \frac{E_t}{E_r} + \frac{E_t}{G_{rt}} = 3$ or if $\frac{E_t}{E_r} = k^2$ we shall obtain: $k^2 - 2 \cdot \mu_{rt} \cdot k^2 = 3 - k^2$, what we have obtained $B_{(1)} = 3 - k^2$ (8) earlier as a result of theoretical studies [1].

Or we may use this formula $\beta_1 = \sqrt{1 + k^2 + \frac{1 + 5 \cdot k^2}{3}} = \frac{2\sqrt{1 + 2 \cdot k^2}}{\sqrt{3}}$, ,

If $k^2 = 1$ we get the same result $\beta_1 = \frac{2\sqrt{1 + 2 \cdot k^2}}{\sqrt{3}}$.

Due to substitution of known ratio so felastic moduli in equation (27), we obtain the same expression (8), which was similarly obtained by S.G. Lekhnitskiy in equation [1] for a curvi linear cylindrically anisotropic orthotropic beam. As to the second elastic parameter $B_{(2)}$, there are no corresponding studies.

It should be noted that to find the ratio β_1, which is necessary for the theoretical study of the stress condition of the brickwork of vaults in the form of cylindrically anisotropic orthotropic bodies, the authors first used the second elastic parameter $B_{(2)}$.

In particular, the elasticity parameter $B_{(2)}$ is recommended to be used to improve the methods for calculating the stress condition and slope part of flat-arched vault. At the same time, orthotropic materials with cylindrical anisotropy can be conventionally divided into two groups.

For the first group of materials that satisfy condition (8), the change inelastic modulus from 0 ° to 90 ° (from the radial to the tangential direction) occurs through an intermediate extreme point when the layers are tilted at an angle of 30 ° to force line.

For the second group, there is no intermediate bending point and elastic modulus changes from 0 ° to 90 ° smoothly.

Fig. 4 shows the obtained numerical values of the distribution of radial σ_r and tangential σ_θ stresses depending on the radius r, considering calculated elastic moduli and ratio β_1.

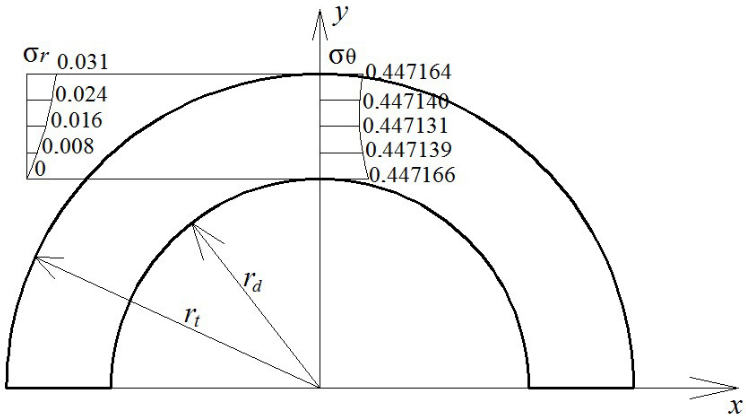

Figure 4. Stress distribution σ_r and σ_θ (MPa) in the crest part of the vault depending on radius *r* at the vault head at angle φ = 0.

4. Conclusion

Using the example of solving the problem of bending a curved beam with cylindrical anisotropy, an approximate method for calculating the stress state of masonry duct vaults is proposed, taking into account the properties of anisotropy.

Findings:

Based on the phenomenological method of rheology and elastic moduli $\left(E_0^{brick}, E_0^{mortar}\right)$, obtained through the experimental strength limits of brick and mortar (R_{brick} = 10.03 MPa, R_{mortar} = 2.14 MPa), the elastic moduli of brickwork in the radial and tangential directions are determined, equal to E_r = 14176 MPa и E_t = 1816 MPa, respectively.

As a result of taking into account the properties of anisotropy at an angle of 90 ° using the MathCad software package, we obtained:

- elasticity parameters: $B_{(1)}$ = 2.872 with coefficient β_1= 2 and $B_{(2)}$ = 0.547 with coefficient β_1 = 1.392;
- modules for parameters $B_{(1)}$ and $B_{(2)}$, equal: for brick E_0^{brick} = 12036 MPa, mortar E_0^{mortar} = 2140 MPa and brickwork E_x = 1816 MPa;
- maximum stress in the locking part, equal to: σ_r = 0.031 MPa and σ_t = 0.447164 MPa.

References

1. Lekhnitskiy, S.G. Anisotropic plates. Moscow, 1957. 463 p. (rus)
2. Kaldar-ool, A-Kh.B., Babanov, V.V., Allahverdov, B.M., Saaya, S.S. Additional load on barrel vaults of architectural monuments. Magazine of Civil Engineering. 2018. 08 (84). Pp. 15–28. DOI: 10.18720/MCE.84.2
3. Kaldar-ool, A-Kh.B., Glukhikh, V.N., Opbul, E.K., Saaya, S.S. Stress condition of brick barrel vaults in view of anisotropic properties. Magazine of Civil Engineering. 2021. 105(5). Article No. 10509. DOI: 10.34910/MCE.105.9Skripchenko, I.V., Bespalov, V.V., Lukichev, S.Y., Zimin, S.S. Unconventional cases of the stone vaults. Construction of Unique Buildings and Structures. 2017. 2 (53). Pp. 87–95.
4. Lalin, V.V., Dmitriev, A.N., Diakov, S.F. Nonlinear deformation and stability of geometrically exact elastic arches. Magazine of Civil Engineering. 2019. 5 (89). S. 39–51. DOI: 10.18720/MCE.89.4
5. Orlovich, R.B., Nowak, R., Vatin, N.I., Bespalov V.V. Natural oscillations of a rectangular plates with two adjacent edges clamped. Magazine of Civil Engineering. 2018. 82 (6). Pp. 95–102. DOI: 10.18720/MCE.82.9
6. Aydin, A.C., Ozkaya, S.G. The finite element analysis of collapse loads of single-spanned historic masonry arch bridges (Ordu, Sarpdere Bridge). Engineering Failure Analysis. 2018. 84. Pp. 131–138. DOI: https://doi.org/10.1016/j.engfailanal.2017.11.002.
7. Varró, R., Bögöly, G., Görög, P. Laboratory and numerical analysis of failure of stone masonry arches with and without reinforcement. Engineering Failure Analysis. 2021. 123 (105272). DOI: https://doi.org/10.1016/j.engfailanal.2021.105272
8. Ferrero C., Calderini C., Portioli F., Roca P. Large displacement analysis of dry-joint masonry arches subject to inclined support movements. Engineering Structures. 2021. 238 (112244). DOI: https://doi.org/10.1016/j.engstruct.2021.112244
9. Boem, I., Gattesco, N. Cyclic behavior of masonry barrel vaults strengthened through Composite Reinforced Mortar, considering the role of the connection with the abutments. Engineering Structures. 2021. 228 (111518). DOI: https://doi.org/10.1016/j.engstruct.2020.111518
10. Dell'Endice, A., Iannuzzo, A., DeJong, M.J., Mele, T.V., Block, P. Modelling imperfections in unreinforced masonry structures: Discrete element simulations and scale model experiments of a pavilion vault. Engineering Structures 2021. 228 (111499). DOI: https://doi.org/10.1016 /j.engstruct.2020.111499
11. Lucchesi, M., Pintucchi, B., Zani, N. The masonry vaults of the cellars in the canal system of Livorno. Procedia Structural Integrity. 2018. 11. Pp. 177–184. DOI: https://doi.org/10.1016/j.prostr.2018.11.024
12. Gaspar, O., Sajtos, I., Sipos, A.A. Friction as a geometric constraint on stereotomy in the minimum thickness analysis of circular and elliptical masonry arches. International Journal of Solids and Structures. 2021. 225 (111056). DOI: https://doi.org/10.1016/j.ijsolstr.2021.111056
13. Ricci, E., Fraddosio, A., Piccioni, M.D., Sacco, E., A new numerical approach for determining optimal thrust curves of masonry arches. European Journal of Mechanics – A/Solids. 2019. 75. Pp. 426–442. DOI: https://doi.org/10.1016/j.euromechsol.2019.02.003
14. Portioli, F.P. Rigid block modelling of historic masonry structures using mathematical programming: a unified formulation for non-linear time history, static pushover and limit equilibrium analysis. Bulletin of Earthquake Engineering. 2020. 18 (1). Pp. 211–239. DOI: https://doi.org/10.1007/s10518-019-00722-0
15. Bertolesi, E., Milani, G., Carozzi, F.G., Poggi, C. Ancient masonry arches and vaults strengthened with TRM, SRG and FRP composites: Numerical analyses. Composite Structures. 2018. 187. Pp. 385–402. DOI: https://doi.org/10.1016/j.compstruct.2017.12.021
16. Orsolya, G., András, A. Sipos & István Sajtos Effect of stereotomy on the lower bound value of minimum thickness of semi-circularmasonry arches. International Journal of Architectural Heritage. 2018. 12. DOI: https://doi.org/10.1080/15583058.2017.1422572
17. Cascini, L., Brando, G., Portioli, F.P., Forgione, R.M., Mazzanti, C., Vasta, M. Force-Based Seismic Evaluation of Retrofitting Interventions of Historic Masonry Castles by 3D Rigid Block Limit Analysis: Experimental and Numerical Analysis in Building Rhabilitation and Strengthening. 2020. 10 (15), 5035. DOI: https://doi.org/10.3390/app10155035
18. Zhang, J.-L., Liu, X., Zhao, J.-B., Yuan, Y., Mang, H. Application of a combined precast and in-situ-cast construction method for large-span underground vaults. Tunnelling and Underground Space Technology. 2021. Vol. 111. 103795. DOI: https://doi.org/10.1016/j.tust.2020.103795
19. Wang, Q., Qin, Q., Jiang, B., Xu, S., Zeng, Z., Luan, Y., Liu, B., Zhang, H. Mechanized construction of fabricated arches for large-diameter tunnels. Automation in Construction, 2021, Vol. 124. 103583. DOI: https://doi.org/10.1016/j.autcon.20-21.103583.
20. Castellano, A., Camassa, D., Fraddosio, A., Scacco, J., Piccioni, M.D., Milani, G. Dynamic damage identification for a full-scale parabolic tuff barrel vault under differential settlements of the supports. Construction and Building Materials. Vol. 291. 2021. 123271. DOI: https://doi.org/10.1016/j.conbuildmat.2021.123271
21. López López, D., Bernat-Maso, E., Gil, L., Roca, P. Experimental testing of a composite structural system using tile vaults as integrated formwork for reinforced concrete. Construction and Building Materials. Vol. 300. 2021. 123974. DOI: https://doi.org/10.1016/j.conbuildmat.2021.123974
22. Russian Set of Rules SP 15.13330.2012. Masonry and reinforced masonry structures. The actual formulation of Construction Norms and Regulations II-22-81*. Moscow: FAU "FCS". 2012. 86 p. (rus)
23. Guidance book for designers of industrial, residential and public buildings. Calculation and theoretical refence book. 2-nd ed., revsed. Edited by A.A. Umanskiy. Moscow. 1972. 599 p. (rus)
24. Residential Building Design Guide. No. 3. The design of residential buildings (to SNiP 2.08.01-85) 2.08.01-85). Moscow. 1989. 305 p. (rus)
25. Glukhikh, V.N., Khrabrova, O.Yu. Bending of Sawn Wood Products Obtained from Conventional Sawing and Parallel to Generatrix Sawing. Architecture and Engineering. 2016. 1 (1). Pp. 4–9. DOI: https://doi.org/10.23968/2500-0055-2016-1-1-4-9
26. Yakovlev, K.P. Brief physical and technical reference book, Moscow, 1960. 411 p. (rus)

Information about authors:

***Anay-Khaak Kaldar-ool**, PhD in Technical Science*
ORCID: https://orcid.org/0000-0002-5304-3747
E-mail: oorzhaka-h@mail.ru

***Eres Opbul**, PhD in Technical Science*
ORCID: https://orcid.org/0000-0002-7796-2350
E-mail: fduecnufce@mail.ru

Received 22.08.2021. Approved after reviewing 31.05.2022. Accepted 31.05.2022.