Magazine of Civil Engineering. 2023. 120(4). Article No. 12010

Magazine of Civil Engineering

ISSN 2712-8172

journal homepage: http://engstroy.spbstu.ru/

Research article

UDC 539.3

DOI: 10.34910/MCE.120.10

Parametric oscillations of a viscous-elastic orthotropic shell of variable thickness

***D.A. Khodzhaev*[1] ✉ *R.A. Abdikarimov*[2], *M. Amabili*[3], *B.A. Normuminov*[1]**

[1] *Tashkent Institute of Irrigation and Agricultural Mechanization Engineers National Research University, Tashkent, Uzbekistan*
[2] *Tashkent Institute of Architecture and Civil Engineering, Tashkent, Uzbekistan*
[3] *McGill University, Montreal, Canada*

✉ *dhodjaev@mail.ru*

Keywords: thin walled shell, viscoelasticity, composite materials, variable thickness, nonlinear vibrations, dynamic stability, Galerkin method, numerical method

Abstract. A solution to the problem of parametric oscillations of a viscous-elastic orthotropic shallow shell of variable thickness is presented. Dynamic loading acts along one side of the shell in the form of a periodic load. Unlike linear problems, the nonlinear problem under consideration could not be solved by applying analytical methods; therefore, approximate methods were used. The mathematical model of the problem is built within the Kirchhoff-Love theory. In this case, tangential inertial forces and geometric non-linearity are taken into account. Deflection and displacements approximation is performed using the Galerkin method in higher order approximations, which allows reducing the problem solution to a system of nonlinear integro-differential equations (IDE) with variable coefficients. The weakly singular Koltunov-Rzhanitsyn kernel with three rheological parameters is used as the relaxation kernel; it describes the viscous-elastic properties of the shallow shell. A numerical method based on the use of quadrature formulas is used to obtain a resolving system of equations for the problem. To obtain numerical results, a computer software was compiled in the Delphi environment for a computational algorithm of the problem solution. The effects of viscous-elastic, orthotropic, nonlinear properties of the shell material, thickness variability, and other physical, mechanical, and geometrical parameters on the dynamic strength of a shallow shell are studied.

Citation: Khodzhaev, D.A., Abdikarimov, R.A., Amabili, M., Normuminov, B.A. Parametric vibrations of a viscoelastic orthotropic shell of variable thickness. Magazine of Civil Engineering. 2023. 120(4). Article no. 12010. DOI: 10.34910/MCE.120.10

1. Introduction

The problem of parametric oscillations of elastic and viscous-elastic thin-wall structures (plates and shells of variable thickness) is one of the most relevant problems in the mechanics of a deformable rigid body. The solution to such problems is of great importance for the modern aerospace industry, rocket technology, and mechanical engineering. Structural elements (plates and shells of variable thickness) can be found in many engineering and building structures, in aviation and motor transport, and in various units.

The first studies devoted to the problem of parametric oscillations of plates and shells of constant thickness, within the framework of the theory of thin plates, include the research work by V.V. Bolotin [1]. To solve problems, he used methods based on the variational approach.

There are a great number of articles in the literature devoted mainly to the dynamic stability and parametric oscillations of elastic thin-wall structures (plates, panels, and shells) under the impact of periodic load.

An analysis of the study of oscillation problems of shells made of various materials, conducted in the period from 2003 to 2013, can be found in [2], [3]. It also contains a review of publications related to parametric oscillations.

Analytical and numerical solutions for different types of structures (plates and shells) were considered in [4].

In [5], the dynamic stability of truncated-conical shells under dynamic axial load was studied. To solve the problem, described by a differential equation of the Mathieu-Hill type, the Galerkin method was used.

Reference [6] is devoted to the dynamic stability of a linearly elastic thin rectangular plate subjected to a bi-axial time-varying load. The differential equation of plate motion was solved using the finite difference method (FDM). To identify domains of dynamic stability, the Mathieu-Hill equation was derived.

The dynamic stability of a viscous-elastic rectangular plate subjected to constant and variable loads in the plane of the plate was considered in [7]. The equation of motion is described by an integro-differential equation with respect to an unknown time function. The effect of the viscous-elastic characteristics of material on the dynamic instability zone was shown.

Reference [8] considers the dynamic instability of laminated non-homogeneous orthotropic truncated-conical shells under periodic axial loading. The problem was reduced to solving the Mathieu equation. Bolotin's method was used to evaluate the behavior of the shell for various parameters.

The dynamic instability of layered composite panels of variable stiffness under non-uniform periodic excitation was studied in [9]. The Ritz method was used to obtain the resolving system of equations for the problem. The domains of dynamic instability were constructed by the Bolotin method.

In [10], the behavior of a footbridge under rhythmic loading was studied. The footbridge was considered a shell of variable thickness. The problem was solved by the FEM. The influence of different mass distributions along the footbridge on its dynamic behavior was analyzed.

In [11], the dynamic stability of a cylindrical shell with linear variable thickness was considered under axial forces and pulsed external pressure. The Bubnov–Galerkin method was used to solve the problem. A resolving system of equations for the problem was derived in the form of an infinite system of homogeneous algebraic equations.

In [12], the dynamic instability of toroidal shells was studied. The Galerkin method was used to obtain a semi-analytical solution to the problem. The results obtained were compared with the ones available in the literature. The effect of various geometric and mechanical parameters on the dynamic instability of shells was studied.

A study of the nonlinear dynamic stability of a cylindrical shell of variable thickness is given in [13]. The equation of motion was derived based on the classical theory of shells in a geometric non-linear formulation. The solution to the equation was obtained by the fourth-order Runge-Kutta method and the Galerkin method. The effect of the characteristics of material and geometrical parameters on the dynamic behavior of a shell was investigated.

In [14], the impact behavior of an elastic spherical shell under step pressure was considered. Initial geometric imperfections were introduced.

The study in [15] concerns the analysis of the nonlinear dynamic behavior and stability of heterogeneous axisymmetric shells of variable thickness. The equation of motion was constructed based on the Kirchhoff–Love hypothesis. The Ritz method was used.

In [16], the dynamic behavior of a three-layer (sandwich) conical shell under the action of a periodic load was studied. At that, various boundary conditions were considered. The problem was reduced to solving an equation of the Mathieu-Hill type, the solution of which was obtained by the Bolotin method. The results obtained were compared with the results obtained by other authors.

The behavior of a sandwich plate under periodic load was considered in [17]. Using the constructed mathematical model of the problem, the effect of various geometric and mechanical parameters of a plate on its dynamic behavior was studied.

The study in [18] is devoted to the dynamic instability of a cylindrical shell made of a thin-walled composite material. The ABAQUS program was used. The influence of the parameters of periodic loading and initial geometric imperfections on the dynamic behavior of the shell was investigated.

In [19], the dynamic stability of sandwich panels under periodic load was considered. The solution to the problem was obtained by reducing the obtained equation of motion to an equation of the Mathieu type and applying the Bolotin method. The effect of different geometric and mechanical parameters of panels on their dynamic behavior was studied.

Reference [20] is devoted to the study of the dynamic stability of cylindrical composite shells under the action of a pulsed loading. The FEM was used with the ABAQUS program. The impact of different parameters on the dynamic behavior of the shell was shown.

In [21], an annular plate under external harmonic excitation was studied. Resolving equations were derived based on the non-linear von Karman theory. New mechanical effects were observed.

A brief analysis of the available scientific publications showed that there are almost no studies of nonlinear oscillations and dynamic stability of thin-wall structures (viscous-elastic plates and shells of variable thickness) [22]–[24]. In the article below, nonlinear parametric oscillations of viscous-elastic shallow shells of variable thickness are numerically studied.

The object of the research is various viscous-elastic thin-wall constructions of variable thickness.

The purpose of the study is to develop effective methods, algorithms and a computer program to evaluate the dynamic behavior of thin-wall constructions, taking into account the viscoelasticity of the material properties and variable thickness.

The following problems were solved to achieve this goal:

- to obtain resolving systems of nonlinear integro-differential equations with singular kernels of viscous-elastic thin-wall constructions of variable thickness under the impact of periodical loads;
- to develop an effective approach to numerical solution, computational algorithm and software products for evaluating the strength of viscous-elastic thin-wall constructions of variable thickness under periodical influences.

2. Methods

A rectangular viscous-elastic shallow shell of variable thickness $h(x,y)$ is considered with an account fort the geometric nonlinearity based on the Kirchhoff-Love hypotheses. Let the shell be dynamically loaded along side a by a periodic load $P(t)=P_0+P_1\cos(\Theta t)$, $P_0,P_1=const,\Theta$ - is the frequency of external periodic load (Fig.1). A coordinate lines x and y of the curvilinear orthogonal coordinate frame is directed along the lines of principal curvatures, and the z-axis - along the internal normal of the middle surface.

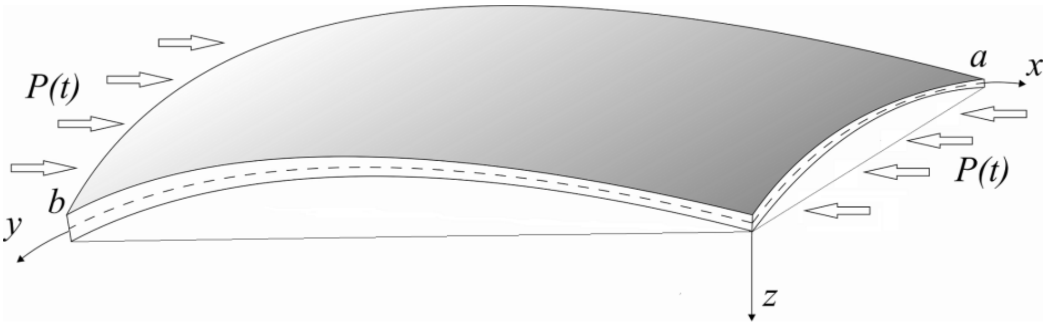


Figure 1. Shallow shell of variable thickness.

The system of equations of motion in the framework of the chosen theory has the following form [25]

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+p_x-\rho h\frac{\partial^2 u}{\partial t^2}=0,\ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+p_y-\rho h\frac{\partial^2 v}{\partial t^2}=0,$$

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 H}{\partial x\partial y}+k_xN_x+k_yN_y+\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)+ \quad (1)$$

$$+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_y\frac{\partial w}{\partial y}\right)+P(t)\frac{\partial^2 w}{\partial x^2}+q-\rho h\frac{\partial^2 w}{\partial t^2}=0,$$

where $k_x = 1/R_1$ and $k_y = 1/R_2$ are the principal curvatures (R_1 and R_2 are the principal radii of curvature) of the shell along the x and y axes, respectively; p_x, p_y and q - external static loads applied to the shell element in directions x, y and z.

The system of equations (1) is supplemented by the corresponding boundary conditions [25], which will be used in the solution to the problems:

1. All edges are simply supported:

at $x = 0, a: u = 0, v = 0, w = 0, M_x = 0$; at $y = 0, b: u = 0, v = 0, w = 0, M_y = 0$.

2. All edges are fixed:

at $x = 0, a: u = 0, v = 0, w = 0, \frac{\partial w}{\partial x} = 0$; at $y = 0, b: u = 0, v = 0, w = 0, \frac{\partial w}{\partial y} = 0$.

3. Two opposite edges are simply supported, the other two edges are fixed:

at $x = 0, a: u = 0, v = 0, w = 0, \frac{\partial w}{\partial x} = 0$; at $y = 0, b: u = 0, v = 0, w = 0, M_y = 0$.

The initial conditions at *t*=0 are as follows:

$$u(x,y,0) = u_0(x,y),\ \dot{u}(x,y,0) = \dot{u}_0(x,y),\ v(x,y,0) = v_0(x,y),\ \dot{v}(x,y,0) = \dot{v}_0(x,y),$$

$$w(x,y,0) = w_0(x,y),\ \dot{w}(x,y,0) = \dot{w}_0(x,y).$$

Here, $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$, $\dot{u}_0(x,y)$, $\dot{v}_0(x,y)$ and $\dot{w}_0(x,y)$ are given functions.

The components of the vector of forces $\{N\} = (N_x, N_y, N_{xy})$ and moments $\{M\} = (M_x, M_y, M_{xy})$ for symmetric structure shells in matrix form can be written as:

$$\{N\} = \{N_x; N_y; N_{xy}\}^T = [C]\cdot\{\varepsilon\},\ \{M\} = \{M_x; M_y; M_{xy}\}^T = [D]\cdot\{\chi\}, \quad (2)$$

here

$$\{\varepsilon\} = (\varepsilon_x, \quad \varepsilon_y, \quad \varepsilon_{xy})^T,\ \{\chi\} = (\chi_x, \quad \chi_y, \quad \chi_{xy})^T,$$

$$\varepsilon_x = \frac{\partial u}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \varepsilon_y = \frac{\partial v}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \varepsilon_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \quad (3)$$

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}, \chi_y = -\frac{\partial^2 w}{\partial y^2}, \chi_{xy} = -\frac{\partial^2 w}{\partial x \partial y}.$$

Stiffness matrices $[C]$ and $[\mathrm{D}]$ have the following form:

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{pmatrix}, \quad (4)$$

where the coefficients of the stiffness matrix C_{ij}, D_{ij} $(ij = 11,22,12,16,26,66)$, depending on the mechanical characteristics of the material and coefficient m are determined as follows:

$$C_{ij} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} B_{ij}\left(1 - \Gamma_{ij}^*\right)dz, \quad D_{ij} = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} B_{ij}\left(1 - \Gamma_{ij}^*\right)z^2 dz, \ (i, j = 1,2,6), \ m = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \rho dz. \quad (5)$$

Here B_{ij} are the stiffness coefficients [26], Γ^*, Γ_{ij}^* – are the integral operators with relaxation kernels $\Gamma(t)$ and $\Gamma_{ij}(t)$, respectively:

$$\Gamma^*\varphi = \int_0^t \Gamma(t-\tau)\varphi(\tau)d\tau, \ \ \Gamma_{ij}^*\varphi = \int_0^t \Gamma_{ij}(t-\tau)\varphi(\tau)d\tau, \ \ i, j = 1,2 .$$

In operator form, the system of equations of motion (1) is written as:

$$L_{11}u + L_{12}v + L_{13}w = -L_{14}w - p_x + \rho h \frac{\partial^2 u}{\partial t^2}, \ \ L_{21}u + L_{22}v + L_{23}w = -L_{24}w - p_y + \rho h \frac{\partial^2 v}{\partial t^2},$$

$$L_{31}u + L_{32}v + L_{33}w = -L_{34}w - q - P(t)\frac{\partial^2 w}{\partial x^2} + \rho h \frac{\partial^2 w}{\partial t^2}. \tag{6}$$

Here u, v and w - are the components of displacement vector $\{U\}$ in the directions of the Ox, Oy , and Oz axes, respectively.

$$L_{11} = C_{11}\frac{\partial^2}{\partial x^2} + 2C_{16}\frac{\partial^2}{\partial x \partial y} + C_{66}\frac{\partial^2}{\partial y^2}, \ \ L_{12} = L_{21} = C_{16}\frac{\partial^2}{\partial x^2} + (C_{12}+C_{66})\frac{\partial^2}{\partial x \partial y} + C_{26}\frac{\partial^2}{\partial y^2},$$

$$L_{13} = -L_{31} = -\left((k_1C_{11} + k_2C_{12})\frac{\partial}{\partial x} + (k_1C_{16} + k_2C_{26})\frac{\partial}{\partial y} \right),$$

$$L_{22} = C_{66}\frac{\partial^2}{\partial x^2} + 2C_{26}\frac{\partial^2}{\partial x \partial y} + C_{22}\frac{\partial^2}{\partial y^2},$$

$$L_{23} = L_{32} = \left((k_1C_{16} + k_2C_{26})\frac{\partial}{\partial x} + (k_1C_{12} + k_2C_{22})\frac{\partial}{\partial y} \right),$$

$$L_{22} = C_{66}\frac{\partial^2}{\partial x^2} + 2C_{26}\frac{\partial^2}{\partial x \partial y} + C_{22}\frac{\partial^2}{\partial y^2},$$

$$L_{33} = D_{11}\frac{\partial^4}{\partial x^4} + 2(\mathrm{D}_{12} + 2D_{66})\frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4}{\partial x \partial y^3} + D_{22}\frac{\partial^4}{\partial y^4} -$$

$$-\left(C_{11}k_x^2 + 2C_{12}k_xk_y + C_{22}k_y^2\right),$$

$$L_{14}(u,v,w,C_{ij}) = \frac{\partial}{\partial x}\left[\frac{1}{2}C_{11}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}C_{12}\left(\frac{\partial w}{\partial y}\right)^2 + C_{16}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] +$$

$$+\frac{\partial}{\partial y}\left[\frac{1}{2}C_{16}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}C_{26}\left(\frac{\partial w}{\partial y}\right)^2 + C_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + \tag{7}$$

$$+\left[\frac{\partial u}{\partial x} - k_xw + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left(\frac{\partial C_{11}}{\partial x} + \frac{\partial C_{16}}{\partial y}\right) + \left[\frac{\partial v}{\partial y} - k_yw + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left(\frac{\partial C_{12}}{\partial x} + \frac{\partial C_{16}}{\partial y}\right) +$$

$$+\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\left(\frac{\partial C_{16}}{\partial x} + \frac{\partial C_{66}}{\partial y}\right),$$

$$L_{24}(u,v,w,C_{ij})=\frac{\partial}{\partial x}\left[\frac{1}{2}C_{16}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}C_{26}\left(\frac{\partial w}{\partial y}\right)^2+C_{66}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]+$$

$$+\frac{\partial}{\partial y}\left[\frac{1}{2}C_{12}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}C_{22}\left(\frac{\partial w}{\partial y}\right)^2+C_{26}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]+$$

$$+\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left(\frac{\partial C_{12}}{\partial y}+\frac{\partial C_{16}}{\partial x}\right)+\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left(\frac{\partial C_{22}}{\partial y}+\frac{\partial C_{26}}{\partial x}\right)+$$

$$+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\left(\frac{\partial C_{26}}{\partial y}+\frac{\partial C_{66}}{\partial x}\right),$$

$$L_{34}(u,v,w,C_{ij},D_{ij})=2\frac{\partial^3 w}{\partial x^3}\left(\frac{\partial D_{11}}{\partial x}+\frac{\partial D_{16}}{\partial y}\right)+2\frac{\partial^3 w}{\partial x\partial y^2}\left(\frac{\partial D_{12}}{\partial x}+3\frac{\partial D_{26}}{\partial y}+2\frac{\partial D_{66}}{\partial x}\right)+$$

$$+2\frac{\partial^3 w}{\partial x^2\partial y}\left(\frac{\partial D_{12}}{\partial y}+3\frac{\partial D_{16}}{\partial x}+2\frac{\partial D_{66}}{\partial y}\right)+2\frac{\partial^3 w}{\partial y^3}\left(\frac{\partial D_{22}}{\partial y}+\frac{\partial D_{26}}{\partial x}\right)+$$

$$+\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 D_{11}}{\partial x^2}+\frac{\partial^2 D_{12}}{\partial y^2}+2\frac{\partial^2 D_{16}}{\partial x\partial y}\right)+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 D_{12}}{\partial x^2}+\frac{\partial^2 D_{22}}{\partial y^2}+2\frac{\partial^2 D_{26}}{\partial x\partial y}\right)+$$

$$+2\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial^2 D_{16}}{\partial x^2}+\frac{\partial^2 D_{26}}{\partial y^2}+2\frac{\partial^2 D_{66}}{\partial x\partial y}\right)+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(k_xC_{11}+k_yC_{12}\right)+$$

$$+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\left(k_xC_{12}+k_yC_{22}\right)+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(k_xC_{16}+k_yC_{26}\right)+$$

$$+\frac{\partial^2 w}{\partial x^2}\left\{C_{11}\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+C_{12}\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+C_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right\}+$$

$$+2\frac{\partial^2 w}{\partial x\partial y}\left\{C_{16}\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+C_{26}\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right\}+$$

$$+\frac{\partial^2 w}{\partial y^2}\left\{C_{12}\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+C_{22}\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+C_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right\}+$$

$$+\frac{\partial w}{\partial x}\left[L_{11}(u)+L_{12}(v)+L_{13}(w)+L_{14}(w)\right]+\frac{\partial w}{\partial y}\left[L_{21}(u)+L_{22}(v)+L_{23}(w)+L_{24}(w)\right].$$

If the shell under consideration has orthotropic properties, then the coefficients are $C_{16}=C_{26}=0$ and $D_{16}=D_{26}=0$. In relationships (2), the stiffness matrices have the following form:

$$C=\begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{pmatrix}, \quad \mathrm{D}=\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{pmatrix}, \tag{8}$$

Here, coefficients C_{ij} and D_{ij} $(ij = 11, 12, 22, 66)$ are expressed in terms of elastic constants E_1, E_2, G_{12}, μ_1, μ_2 as follows:

$$C_{11} = hB_{11}\left(1-\varGamma_{11}^*\right) = \frac{hE_1\left(1-\varGamma_{11}^*\right)}{1-\mu_1\mu_2}, \;\; C_{22} = hB_{22}\left(1-\varGamma_{22}^*\right) = \frac{hE_2\left(1-\varGamma_{22}^*\right)}{1-\mu_1\mu_2},$$

$$C_{12} = B_{12}\left(1-\varGamma_{12}^*\right)h = \frac{\mu_2 E_1\left(1-\varGamma_{12}^*\right)h}{1-\mu_1\mu_2}, \; C_{66} = B_{66}\left(1-\varGamma_{66}^*\right)h = hG_{12}\left(1-\varGamma_{66}^*\right),$$

$$D_{11} = B_{11}\left(1-\varGamma_{11}^*\right)h^3 = \frac{E_1\left(1-\varGamma_{11}^*\right)h^3}{12(1-\mu_1\mu_2)}, \;\; D_{22} = B_{22}\left(1-\varGamma_{22}^*\right)h^3 = \frac{E_2\left(1-\varGamma_{22}^*\right)h^3}{12(1-\mu_1\mu_2)},$$

$$D_{12} = B_{12}\left(1-\varGamma_{12}^*\right)h^3 = \frac{\mu_2 E_1\left(1-\varGamma_{12}^*\right)h^3}{12(1-\mu_1\mu_2)}, \;\; C_{66} = B_{66}\left(1-\varGamma_{66}^*\right)h^3 \frac{G_{12}}{12}h^3. \tag{9}$$

Here $E_1, E_2 -$ are the moduli of elasticity in the direction of the x and y axes; G_{12} is the shear modulus; $\mu_1, \mu_2 -$ are the Poisson's ratios.

If the shell has isotropic properties ($E_1 = E_2, \mu_1 = \mu_2$), then the elements of the stiffness matrix take a simpler form with two elastic constants E (modulus of elasticity) and μ (Poisson's ratio):

$$C_{11} = C_{22} = B\left(1-\varGamma^*\right)h = \frac{E\left(1-\varGamma^*\right)h}{1-\mu^2}, \;\; C_{12} = \mu B\left(1-\varGamma^*\right)h = \frac{\mu E h}{1-\mu^2},$$

$$C_{66} = \frac{(1-\mu)}{2}B\left(1-\varGamma^*\right)h = \frac{E\left(1-\varGamma^*\right)h}{2(1+\mu)}, \;\; D_{11} = D_{22} = B\left(1-\varGamma^*\right)\frac{h^3}{12} = \frac{E\left(1-\varGamma^*\right)h^3}{12(1-\mu^2)},$$

$$D_{12} = \mu B\left(1-\varGamma^*\right)\frac{h^3}{12} = \frac{\mu E\left(1-\varGamma^*\right)h^3}{12(1-\mu^2)}, \; D_{66} = \frac{1-\mu}{2}B\left(1-\varGamma^*\right)\frac{h^3}{12} = \frac{E\left(1-\varGamma^*\right)h^3}{24(1+\mu)}. \tag{10}$$

By introducing into equation (6) the following dimensionless quantities

$$\bar{u} = \frac{u}{h_0}; \; \bar{v} = \frac{v}{h_0}; \; \bar{w} = \frac{w}{h_0}; \; \bar{x} = \frac{x}{a}; \; \bar{y} = \frac{y}{b}; \; \bar{t} = \omega t; \; \bar{h} = \frac{h}{h_0}; \; \lambda = \frac{a}{b}; \; \delta = \frac{b}{h_0}; \; \bar{k}_x = \frac{a^2}{h_0 R_1};$$

$$\bar{k}_y = \frac{b^2}{h_0 R_2}; \; \bar{q} = \frac{q}{\sqrt{E_1 E_2}}\left(\frac{b}{h_0}\right)^4; \; \overline{p_x} = \frac{p_x}{\sqrt{E_1 E_2}}; \; \overline{p_y} = \frac{p_y}{\sqrt{E_1 E_2}}; \; \bar{\Theta} = \frac{\Theta}{\omega}; \; \delta_0 = \frac{P_0}{P_{cr}}; \; \delta_1 = \frac{P_1}{P_{cr}}$$

and taking into account relationship (9) for orthotropic shells and keeping the previous notation, we obtain a dimensionless system of nonlinear integro-differential equations for problems of parametric oscillations of a viscous-elastic orthotropic shallow shell of variable thickness. Here, operators L_{ij} take the following form:

$$L_{11}(u) = h\Delta\left(1-\Gamma_{11}^*\right)\frac{\partial^2 u}{\partial x^2} + \lambda^2 h\left(1-\mu_1\mu_2\right)g\left(1-\Gamma_{11}^*\right)\frac{\partial^2 u}{\partial y^2} + \frac{\partial h}{\partial x}\Delta\left(1-\Gamma_{11}^*\right)\frac{\partial u}{\partial x} +$$

$$+\lambda^2\frac{\partial h}{\partial y}(1-\mu_1\mu_2)g\left(1-\Gamma^*\right)\frac{\partial u}{\partial y},$$

$$L_{12}(v)=\lambda h\left[\mu_2\Delta\left(1-\Gamma_{12}^*\right)+(1-\mu_1\mu_2)g\left(1-\Gamma^*\right)\right]\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial h}{\partial x}\mu_2\Delta\left(1-\Gamma_{12}^*\right)\frac{\partial v}{\partial y}+$$

$$\lambda\frac{\partial h}{\partial y}(1-\mu_1\mu_2)g\left(1-\Gamma^*\right)\frac{\partial v}{\partial x},$$

$$L_{13}(w)=h\left[\frac{k_x\Delta}{\lambda\delta}\left(1-\Gamma_{11}^*\right)+\frac{\lambda k_y}{\delta}\mu_2\Delta\left(1-\Gamma_{12}^*\right)\right]\frac{\partial w}{\partial x}-\frac{\partial h}{\partial x}\left[\left[\frac{k_x\Delta}{\lambda\delta}\left(1-\Gamma_{11}^*\right)+\frac{\lambda k_y\mu_2\Delta}{\delta}\left(1-\Gamma_{12}^*\right)w\right],\right.$$

$$L_{14}(w)=\frac{h\Delta}{\lambda\delta}\left(1-\Gamma_{11}^*\right)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\lambda h}{\delta}\left[\mu_2\Delta\left(1-\Gamma_{12}^*\right)+(1-\mu_1\mu_2)g\left(1-\Gamma^*\right)\right]\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+$$

$$+(1-\mu_1\mu_2)\frac{\lambda gh}{\delta}\left(1-\Gamma^*\right)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{\partial h}{\partial x}\left[\frac{\Delta}{2\lambda\delta}\left(1-\Gamma_{11}^*\right)\left(\frac{\partial w}{\partial x}\right)^2+\right.$$

$$\left.+\frac{\lambda\mu_2\Delta}{2\delta}\left(1-\Gamma_{12}^*\right)\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{\partial h}{\partial y}(1-\mu_1\mu_2)\frac{\lambda g}{\delta}\left(1-\Gamma^*\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

$$L_{21}(u)=\frac{h}{\lambda}\left[\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)+(1-\mu_1\mu_2)g\left(1-\Gamma^*\right)\right]\frac{\partial^2 u}{\partial x\partial y}+$$

$$+(1-\mu_1\mu_2)\frac{g}{\lambda}\frac{\partial h}{\partial x}\left(1-\Gamma^*\right)\frac{\partial u}{\partial y}+\frac{\mu_1}{\lambda\Delta}\frac{\partial h}{\partial y}\left(1-\Gamma_{21}^*\right)\frac{\partial u}{\partial x},$$

$$L_{22}(м)=\frac{h}{\Delta}\left(1-\Gamma_{22}^*\right)\left(\frac{\partial^2 v}{\partial y^2}\right)+(1-\mu_1\mu_2)\frac{gh}{\lambda^2}\left(1-\Gamma^*\right)\frac{\partial^2 v}{\partial x^2}+$$

$$+(1-\mu_1\mu_2)\frac{g}{\lambda^2}\frac{\partial h}{\partial x}\left(1-\Gamma^*\right)\frac{\partial v}{\partial x}+\frac{1}{\Delta}\frac{\partial h}{\partial y}\left(1-\Gamma_{22}^*\right)\frac{\partial v}{\partial y},$$

$$L_{23}(w)=h\left[\frac{k_x}{\lambda^2\delta}\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)+\frac{k_y}{\delta\Delta}\left(1-\Gamma_{22}^*\right)\right]\frac{\partial w}{\partial y}+\frac{\partial h}{\partial y}\left[\frac{k_y}{\delta\Delta}\left(1-\Gamma_{22}^*\right)+\frac{k_x}{\lambda^2\delta}\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)w\right],$$

$$L_{24}(w)=h\left\{\frac{1}{\delta\Delta}\left(1-\Gamma_{22}^*\right)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\left[\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)+(1-\mu_1\mu_2)\frac{g}{\lambda^2\delta}\left(1-\Gamma^*\right)\right]\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\right.$$

$$\left.+(1-\mu_1\mu_2)\frac{g}{\lambda^2\delta}\left(1-\Gamma^*\right)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}\right\}+\frac{\partial h}{\partial x}(1-\mu_1\mu_2)\frac{g}{\lambda^2\delta}\left(1-\Gamma^*\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+$$

$$+\frac{\partial h}{\partial y}\left[\frac{1}{2\delta\Delta}\left(1-\Gamma_{22}^*\right)\left(\frac{\partial w}{\partial y}\right)^2+\frac{\mu_1}{2\lambda^2\delta\Delta}\left(1-\Gamma_{21}^*\right)\left(\frac{\partial w}{\partial x}\right)^2\right],$$

$$L_{31}(u)=-12h\left\{\left[\lambda\delta\Delta k_x\left(1-\Gamma_{11}^*\right)+\frac{\lambda^3\delta\mu_1 k_y}{\Delta}\left(1-\Gamma_{21}^*\right)\right]\frac{\partial u}{\partial x}+\right.$$

$$+\frac{\partial h}{\partial x}\lambda\delta\Delta\left(1-\Gamma_{11}^{*}\right)\frac{\partial u}{\partial x}+\frac{\partial h}{\partial y}\left(1-\mu_1\mu_2\right)\lambda^3\delta g\left(1-\Gamma^{*}\right)\frac{\partial u}{\partial y}+$$

$$+\frac{\partial h}{\partial x}\left(1-\mu_1\mu_2\right)\lambda^3\delta g\left(1-\Gamma^{*}\right)\frac{\partial u}{\partial y}+\frac{\partial h}{\partial y}\frac{\mu_1\lambda^3\delta}{\Delta}\left(1-\Gamma_{21}^{*}\right)\frac{\partial u}{\partial x},$$

$$L_{32}(v)=\left[\lambda^2\delta\mu_2\Delta k_x\left(1-\Gamma_{12}^{*}\right)+\frac{\lambda^4\delta k_y}{\Delta}\left(1-\Gamma_{22}^{*}\right)\right]\frac{\partial v}{\partial y}-$$

$$+\frac{\partial h}{\partial x}\mu_2\lambda^2\delta\Delta\left(1-\Gamma_{12}^{*}\right)\frac{\partial v}{\partial y}+\frac{\partial h}{\partial y}\left(1-\mu_1\mu_2\right)\lambda^2\delta g\left(1-\Gamma^{*}\right)\frac{\partial v}{\partial x}+$$

$$+\frac{\partial h}{\partial x}\left(1-\mu_1\mu_2\right)\lambda^2\delta g\left(1-\Gamma^{*}\right)\frac{\partial v}{\partial x}+\frac{\partial h}{\partial y}\frac{\lambda^4\delta}{\Delta}\left(1-\Gamma_{22}^{*}\right)\frac{\partial v}{\partial y},$$

$$L_{33}(w)=h^3\left\{\Delta\left(1-\Gamma_{11}^{*}\right)\frac{\partial^4 w}{\partial x^4}+\left[4\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^{*}\right)+\mu_2\Delta\left(1-\Gamma_{12}^{*}\right)+\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^{*}\right)\right]\lambda^2\frac{\partial^4 w}{\partial x^2\partial y^2}+\right.$$

$$\left.+\frac{\lambda^4}{\Delta}\left(1-\Gamma_{22}^{*}\right)\frac{\partial^4 w}{\partial y^4}\right\}+3\left[2h\left(\frac{\partial h}{\partial x}\right)^2+h^2\frac{\partial^2 h}{\partial x^2}\right]\left[\Delta\left(1-\Gamma_{11}^{*}\right)\frac{\partial^2 w}{\partial x^2}+\lambda^2\mu_2\Delta\left(1-\Gamma_{12}^{*}\right)\frac{\partial^2 w}{\partial y^2}\right]+$$

$$+6h^2\frac{\partial h}{\partial x}\left\{\Delta\left(1-\Gamma_{11}^{*}\right)\frac{\partial^3 w}{\partial x^3}+\left[\mu_2\Delta\left(1-\Gamma_{12}^{*}\right)+2\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^{*}\right)\right]\lambda^2\frac{\partial^3 w}{\partial x\partial y^2}\right\}+$$

$$+6h^2\frac{\partial h}{\partial y}\left\{\frac{1}{\Delta}\left(1-\Gamma_{22}^{*}\right)\lambda^4\frac{\partial^3 w}{\partial y^3}+\left[\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^{*}\right)+2\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^{*}\right)\right]\lambda^2\frac{\partial^3 w}{\partial x^2\partial y}\right\}+$$

$$+3\left[2h\left(\frac{\partial h}{\partial y}\right)^2+h^2\frac{\partial^2 h}{\partial y^2}\right]\left[\frac{1}{\Delta}\left(1-\Gamma_{22}^{*}\right)\lambda^4\frac{\partial^2 w}{\partial y^2}+\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^{*}\right)\lambda^2\frac{\partial^2 w}{\partial x^2}\right]+$$

$$+12\left[2h\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}+h^2\frac{\partial^2 h}{\partial x\partial y}\right]\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^{*}\right)\lambda^2\frac{\partial^2 w}{\partial x\partial y}-$$

$$-12h\left\{\Delta k_x^2\left(1-\Gamma_{11}^{*}\right)+\lambda^2 k_x k_y\left[\mu_2\Delta\left(1-\Gamma_{12}^{*}\right)+\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^{*}\right)\right]+\frac{\lambda^4 k_y^2}{\Delta}\left(1-\Gamma_{22}^{*}\right)\right\}w-$$

$$-\frac{\partial h}{\partial x}\left[k_x\Delta\left(1-\Gamma_{11}^{*}\right)+\lambda^2 k_y\mu_2\Delta\left(1-\Gamma_{12}^{*}\right)\right]w+\frac{\partial h}{\partial y}\left[\frac{\lambda^4 k_y}{\Delta}\left(1-\Gamma_{22}^{*}\right)+\frac{\lambda^2\mu_1 k_x}{\Delta}\left(1-\Gamma_{21}^{*}\right)\right]w,$$

$$L_{34}(u,v,w)=\frac{1}{2}\left[k_x\Delta\left(1-\Gamma_{11}^{*}\right)+\frac{\lambda^2\mu_1 k_y}{\Delta}\left(1-\Gamma_{21}^{*}\right)\right]\left(\frac{\partial w}{\partial x}\right)^2+$$

$$+\frac{1}{2}\left[\lambda^2 k_x \mu_2 \Delta\left(1-\Gamma_{12}^*\right)+\frac{\lambda^4 k_y}{\Delta}\left(1-\Gamma_{22}^*\right)\right]\left(\frac{\partial w}{\partial y}\right)^2\Bigg\}-$$

$$-12\frac{\partial w}{\partial x}\Bigg\{h\Bigg\{\Delta\left(1-\Gamma_{11}^*\right)\left(\lambda\delta\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)-\left[k_x\Delta\left(1-\Gamma_{11}^*\right)+\lambda^2\mu_2\Delta k_y\left(1-\Gamma_{12}^*\right)\right]\frac{\partial w}{\partial x}+$$

$$+\left[\mu_2\Delta\left(1-\Gamma_{12}^*\right)+\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^*\right)\right]\left(\lambda^2\delta\frac{\partial^2 v}{\partial x\partial y}+\lambda^2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)+ \tag{11}$$

$$+\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^*\right)\left(\lambda^3\delta\frac{\partial^2 u}{\partial y^2}+\lambda^2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)\Bigg\}-$$

$$-12\frac{\partial^2 w}{\partial x^2}h\Bigg\{\Delta\left(1-\Gamma_{11}^*\right)\left[\lambda\delta\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\mu_2\Delta\left(1-\Gamma_{12}^*\right)\left[\lambda^2\delta\frac{\partial v}{\partial y}+\frac{\lambda^2}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]-$$

$$-\left[k_x\Delta\left(1-\Gamma_{11}^*\right)+\lambda^2 k_y\mu_2\Delta\left(1-\Gamma_{12}^*\right)\right]w\Bigg\}-12\frac{\partial w}{\partial y}\Bigg\{h\Bigg\{\frac{1}{\Delta}\left(1-\Gamma_{22}^*\right)\left(\lambda^4\delta\frac{\partial^2 v}{\partial y^2}+\lambda^4\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right)-$$

$$-\left[\frac{\lambda^2\mu_1 k_x}{\Delta}\left(1-\Gamma_{21}^*\right)+\frac{\lambda^4 k_y}{\Delta}\left(1-\Gamma_{22}^*\right)\right]\frac{\partial w}{\partial y}+\left[\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)+\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^*\right)\right]\times$$

$$\times\left(\lambda^3\delta\frac{\partial^2 u}{\partial x\partial y}+\lambda^2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right)+\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^*\right)\left(\lambda^2\delta\frac{\partial^2 v}{\partial x^2}+\lambda^2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)\Bigg\}+$$

$$-12\frac{\partial^2 w}{\partial y^2}h\Bigg\{\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)\left[\lambda^3\delta\frac{\partial u}{\partial x}+\frac{\lambda^2}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+$$

$$+\frac{1}{\Delta}\left(1-\Gamma_{22}^*\right)\left[\lambda^4\delta\frac{\partial v}{\partial y}+\frac{\lambda^4}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]-\left[\frac{\lambda^2\mu_1 k_x}{\Delta}\left(1-\Gamma_{21}^*\right)+\frac{\lambda^4 k_y}{\Delta}\left(1-\Gamma_{22}^*\right)\right]w\Bigg\}-$$

$$-24\frac{\partial^2 w}{\partial x\partial y}h\left(1-\mu_1\mu_2\right)g\left(1-\Gamma^*\right)\left(\lambda^3\delta\frac{\partial u}{\partial y}+\lambda^2\delta\frac{\partial v}{\partial x}+\lambda^2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)+$$

$$+\frac{\partial h}{\partial x}\Bigg\{\Delta\left(1-\Gamma_{11}^*\right)\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\mu_2\Delta\left(1-\Gamma_{12}^*\right)\left[\frac{\lambda^2}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\Bigg\}+$$

$$+\left(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}\right)\left(1-\mu_1\mu_2\right)\lambda^2 g\left(1-\Gamma^*\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial h}{\partial y}\Bigg\{\frac{1}{\Delta}\left(1-\Gamma_{22}^*\right)\left[\frac{\lambda^4}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{\mu_1}{\Delta}\left(1-\Gamma_{21}^*\right)\left[\frac{\lambda^2}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

,

here $P_{cr}=\frac{\pi^2}{3\left(1-\mu_1\mu_2\right)}\sqrt{E_1E_2}\left(\frac{h_0}{b}\right)^2$ is the static critical load; $\omega=\sqrt{\pi^2\sqrt{E_1E_2}\,h_0^2P_{cr}^*/(\rho b^4)}$ is the frequency of the fundamental tone of oscillations; $P_{cr}^{*}=\frac{P_{cr}}{\sqrt{E_1E_2}(b/h_0)^2}=\frac{\pi^2}{3(1-\mu_1\mu_2)}$; $\Delta=\sqrt{E_1/E_2}$

.

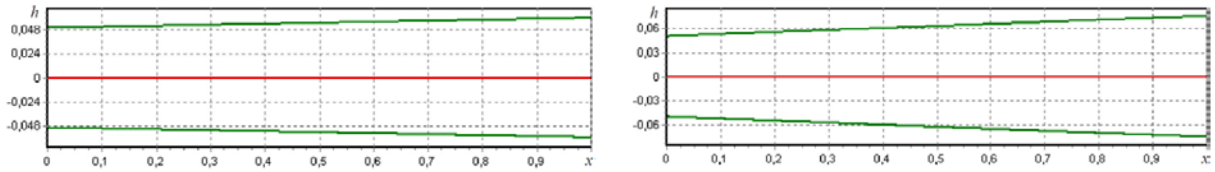
The system of equations (6), (11) with the corresponding boundary and initial conditions describes the motion of a viscous-elastic orthotropic shallow shell of variable thickness under a periodic load $P(t)=P_0+P_1\cos(\Theta t)$.

In calculations, the singular Koltunov-Rzhanitsin kernels [27] are used as relaxation kernels:

$$\Gamma(t)=Ae^{-\beta t}t^{\alpha-1},(0<\alpha<1),\ \Gamma_{ij}(\mathrm{t})=A_{ij}e^{-\beta_{ij}t}t^{\alpha_{ij}-1},\left(0<\alpha_{ij}<1\right) \qquad (12)$$

Let the shell thickness change following the law $h(x)=\frac{1}{2}h_0(1+\alpha * x)$, i.e., it leads to a linear increase in the shell thickness (Fig.2).

Here, α^* is the parameter characterizing the thickness variability; h_0 is the shell thickness corresponding to $\alpha^*=0$.



a) **b)**

Figure 2. Change in the shell thickness depending on the value of parameter α^*:

a) $\alpha^*=0.2$**; b)** $\alpha^*=0.5$

The solution to the obtained IDE system that satisfies the boundary conditions of the problem is sought with respect to the displacements *u* and *v*, and the deflection *w* in the form

$$u(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nm}(t)\phi_{nm}(x,y),\ v(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}v_{nm}(t)\varphi_{nm}(x,y),$$

$$w(x,y,t)=\sum_{n=1}^{N}\sum_{m=1}^{M}w_{nm}(t)\psi_{nm}(x,y), \qquad (13)$$

where $u_{nm}=u_{nm}(t)$, $v_{nm}=v_{nm}(t)$, $w_{nm}=w_{nm}(t)$ - unknown functions of time; $\phi_{nm}(x,y)$, $\varphi_{nm}(x,y)$, $\psi_{nm}(x,y)$, $n=1,2,...,N$; $m=1,2,...,M$ - coordinate functions that satisfy the given boundary conditions of the problem.

Substituting (13) into the system of equations (6), (11) and performing the Bubnov-Galerkin procedure, we obtain the following system of basic resolving nonlinear IDEs:

$$\sum_{n=1}^{N}\sum_{m=1}^{M}a_{klnm}\ddot{u}_{nm}-\eta_1\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{11}^{*}\right)d_{1klnm}+\left(1-\Gamma^{*}\right)d_{2klnm}\right]u_{nm}+\right.\right.$$

$$+\left[\left(1-\Gamma_{12}^{*}\right)d_{3klnm}+\left(1-\Gamma^{*}\right)d_{4klnm}\right]v_{nm}+\left[\left(1-\Gamma_{11}^{*}\right)d_{5klnm}+\left(1-\Gamma_{12}^{*}\right)d_{6klnm}\right]w_{nm}\Big\}+$$

$$\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma_{11}^{*}\right)d_{7klnmij}+\left(1-\Gamma_{12}^{*}\right)d_{8klnmij}+\left(1-\Gamma^{*}\right)d_{9klnmij}\right]\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)\right\}=0,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}\ddot{v}_{nm} - \eta_2\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{21}^{*}\right)e_{1klnm} + \left(1-\Gamma^{*}\right)e_{2klnm}\right]u_{nm} + \right.\right.$$

$$+\left[\left(1-\Gamma_{22}^{*}\right)e_{3klnm} + \left(1-\Gamma^{*}\right)e_{4klnm}\right]v_{nm} + \left[\left(1-\Gamma_{21}^{*}\right)e_{5klnm} + \left(1-\Gamma_{22}^{*}\right)e_{6klnm}\right]w_{nm}\Big\} + \tag{14}$$

$$\left. + \sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma_{22}^{*}\right)e_{7klnmij} + \left(1-\Gamma_{21}^{*}\right)e_{8klnmij} + \left(1-\Gamma^{*}\right)e_{9klnmij}\right]\left(w_{nm}w_{ij} - w_{0nm}w_{0ij}\right)\right\} = 0,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}\ddot{w}_{nm} + \eta_3\sum_{n=1}^{N}\sum_{m=1}^{M} p_{klnm}^{2}\left(1-2\mu_{klnm}\cos\Theta t\right)w_{nm} -$$

$$-\eta_3\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{11}^{*}\right)f_{1klnm} + \left(1-\Gamma_{21}^{*}\right)f_{2klnm}\right]u_{nm} + \left[\left(1-\Gamma_{12}^{*}\right)f_{3klnm} + \left(1-\Gamma_{22}^{*}\right)f_{4klnm}\right]v_{nm} + \right.\right.$$

$$\left.\left. +\left[\Gamma_{11}^{*}f_{5klnm} + \Gamma_{12}^{*}f_{6klnm} + \Gamma_{22}^{*}f_{7klnm} + \Gamma_{21}^{*}f_{8klnm} + \Gamma^{*}f_{9klnm}\right]w_{0nm}\right\}\right\} -$$

$$-\eta_3\left\{\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} w_{nm}\left\{\left[\left(1-\Gamma_{11}^{*}\right)\xi_{1klnmij} + \left(1-\Gamma_{21}^{*}\right)\xi_{2klnmij} + \left(1-\Gamma^{*}\right)\xi_{3klnmij}\right]u_{ij} + \right.\right.$$

$$\left. +\left[\left(1-\Gamma_{22}^{*}\right)\xi_{4klnmij} + \left(1-\Gamma_{12}^{*}\right)\xi_{5klnmij} + \left(1-\Gamma^{*}\right)\xi_{6klnmij}\right]v_{ij}\right\} +$$

$$+\left[\left(1-\Gamma_{11}^{*}\right)\xi_{7klnmij} + \left(1-\Gamma_{12}^{*}\right)\xi_{8klnmij} + \left(1-\Gamma_{22}^{*}\right)\xi_{9klnmij} + \left(1-\Gamma_{21}^{*}\right)\xi_{10klnmij}\right]\left(w_{ij} - w_{0ij}\right)\Big\} +$$

$$+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left\{\left(1-\Gamma_{11}^{*}\right)g_{1klnmijrs} + \left(1-\Gamma_{12}^{*}\right)g_{2klnmijrs} + \left(1-\Gamma_{21}^{*}\right)g_{3klnmijrs} + \left(1-\Gamma_{22}^{*}\right)g_{4klnmijrs}\right\}\left(w_{nm}w_{ij} - w_{0nm}w_{0ij}\right) +$$

$$+\sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M} w_{nm}\left\{\left(1-\Gamma_{11}^{*}\right)g_{5klnmijrs} + \left(1-\Gamma_{12}^{*}\right)g_{6klnmijrs} + \left(1-\Gamma_{22}^{*}\right)g_{7klnmijrs} + \right.$$

$$\left.\left. +\left(1-\Gamma_{21}^{*}\right)g_{8klnmijrs} + \left(1-\Gamma^{*}\right)g_{9klnmijrs}\right\}\left(w_{ij}w_{rs} - w_{0ij}w_{0rs}\right)\right\} = 12\eta_3\left(1-\mu_1\mu_2\right)\lambda^4 q_{kl},$$

$$u_{nm}(0) = u_{0nm},\ \dot{u}_{nm}(0) = \dot{u}_{0nm},\ v_{nm}(0) = v_{0nm},\ \dot{v}_{nm}(0) = \dot{v}_{0nm},$$

$$w_{nm}(0) = w_{0nm},\ \dot{w}_{nm}(0) = \dot{w}_{0nm},\ k = 1,2,\ldots,N;\ l = 1,2,\ldots,M,$$

where the constant coefficients included in this system are related to coordinate functions and their derivatives and have the following form:

$$a_{klnm} = \int_0^1\int_0^1 h\phi_{nm}\phi_{kl}\,dxdy;$$

$$d_{1klnm} = \int_0^1\int_0^1 \Delta\left(h\phi''_{nm,xx} + h'_x\phi'_{nm,x}\right)\phi_{kl}\,dxdy;$$

$$d_{2klnm} = \int_0^1\int_0^1 \left(1-\mu_1\mu_2\right)g\lambda^2\left(h\phi''_{nm,yy} + h'_y\phi'_{nm,y}\right)\phi_{kl}\,dxdy;$$

$$d_{3klnm}=\int_0^1\int_0^1 \mu_2\Delta\lambda\left(h\varphi''_{nm,xy}+h'_x\varphi'_{nm,y}\right)\phi_{kl}\,dxdy\,;$$

$$d_{4klnm}=\int_0^1\int_0^1 (1-\mu_1\mu_2)g\lambda\left(h\varphi''_{nm,xy}+h'_y\varphi'_{nm,x}\right)\phi_{kl}\,dxdy\,;$$

$$d_{5klnm}=-\int_0^1\int_0^1 \frac{\Delta k_x}{\lambda\delta}\left(h\psi'_{nm,x}+h'_x\psi_{nm}\right)\phi_{kl}\,dxdy\,;$$

$$d_{6klnm}=-\int_0^1\int_0^1 \frac{\Delta k_y\mu_2\lambda}{\delta}\left(h\psi'_{nm,x}+h'_x\psi_{nm}\right)\phi_{kl}\,dxdy\,;$$

$$d_{7klnmij}=\int_0^1\int_0^1 \frac{\Delta}{\lambda\delta}\left(h\psi'_{nm,x}\psi''_{ij,xx}+\frac{1}{2}h'_x\psi'_{nm,x}\psi'_{ij,x}\right)\phi_{kl}\,dxdy\,;$$

$$d_{8klnmij}=\int_0^1\int_0^1 \frac{\mu_2\Delta\lambda}{\delta}\left(h\psi'_{nm,y}\psi''_{ij,xy}+\frac{1}{2}h'_x\psi'_{nm,y}\psi'_{ij,y}\right)\phi_{kl}\,dxdy\,;$$

$$d_{9klnmij}=\int_0^1\int_0^1 (1-\mu_1\mu_2)g\,\frac{\lambda}{\delta}\left(h\psi'_{nm,y}\psi''_{ij,xy}+h\psi'_{nm,x}\psi''_{ij,yy}+h'_y\psi'_{nm,x}\psi'_{ij,y}\right)\phi_{kl}\,dxdy\,;$$

$$b_{klnm}=\int_0^1\int_0^1 h\varphi_{nm}\varphi_{kl}\,dxdy\,;$$

$$e_{1klnm}=\int_0^1\int_0^1 \frac{\mu_1}{\Delta\lambda}\left(h\phi''_{nm,xy}+h'_y\phi'_{nm,x}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{2klnm}=\int_0^1\int_0^1 \frac{(1-\mu_1\mu_2)g}{\lambda}\left(h\phi''_{nm,xy}+h'_x\phi'_{nm,y}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{3klnm}=\int_0^1\int_0^1 \frac{1}{\Delta}\left(h\varphi''_{nm,yy}+h'_y\varphi'_{nm,y}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{4klnm}=\int_0^1\int_0^1 \frac{(1-\mu_1\mu_2)g}{\lambda^2}\left(h\varphi''_{nm,xx}+h'_x\varphi'_{nm,x}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{5klnm}=-\int_0^1\int_0^1 \frac{k_x\mu_1}{\lambda^2\delta\Delta}\left(h\psi'_{nm,y}+h'_y\psi_{nm}\right)\varphi_{kl}\,dxdy\,;\quad e_{6klnm}=-\int_0^1\int_0^1 \frac{k_y}{\delta\Delta}\left(h\psi'_{nm,y}+h'_y\psi_{nm}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{7klnmij}=\int_0^1\int_0^1 \frac{1}{\delta\Delta}\left(h\psi'_{nm,y}\psi''_{ij,yy}+\frac{1}{2}h'_y\psi'_{nm,y}\psi'_{ij,y}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{8klnmij}=\int_0^1\int_0^1 \frac{\mu_1}{\lambda^2\delta\Delta}\left(h\psi'_{nm,x}\psi''_{ij,xy}+\frac{1}{2}h'_y\psi'_{nm,x}\psi'_{ij,x}\right)\varphi_{kl}\,dxdy\,;$$

$$e_{9klnmij}=\int_0^1\int_0^1 \frac{(1-\mu_1\mu_2)g}{\lambda^2\delta}\left(h\psi'_{nm,x}\psi''_{ij,xy}+h\psi'_{nm,y}\psi''_{ij,xx}+h'_x\psi'_{nm,x}\psi'_{ij,y}\right)\varphi_{kl}\,dxdy\,;$$

$$c_{klnm}=\int_0^1\int_0^1 h\psi_{nm}\psi_{kl}\,dxdy\,;$$

$$f_{1klnm} = -12\int_0^1\int_0^1 \lambda\delta\Delta k_x h\phi'_{nm,x}\psi_{kl}dxdy\,; \quad f_{2klnm} = -12\int_0^1\int_0^1 \frac{\lambda^3\delta\mu_1 k_y}{\Delta} h\phi'_{nm,x}\psi_{kl}dxdy\,;$$

$$f_{3klnm} = -12\int_0^1\int_0^1 \lambda^2\delta\mu_2\Delta k_x h\varphi'_{nm,y}\psi_{kl}dxdy\,; \quad f_{4klnm} = -12\int_0^1\int_0^1 \frac{\lambda^4\delta k_y}{\Delta} h\varphi'_{nm,y}\psi_{kl}dxdy\,;$$

$$f_{5klnm} = \int_0^1\int_0^1 \Delta\left(h^3\psi^{IV}_{nm,xxxx} + 3\left[2h(h'_x)^2 + h^2h''_{xx}\right]\psi''_{nm,xx} + 6h^2h'_x\psi'''_{nm,xxx} + 12k_x^2h\psi_{nm}\right)\psi_{kl}dxdy\,;$$

$$f_{6klnm} = \int_0^1\int_0^1 \mu_2\Delta\lambda^2\left(h^3\psi^{IV}_{nm,xxyy} + 3\left[2h(h'_x)^2 + h^2h''_{xx}\right]\psi''_{nm,yy} + 6h^2h'_x\psi'''_{nm,xyy} + 12k_xk_yh\psi_{nm}\right)\psi_{kl}dxdy\,;$$

$$f_{7klnm} = \int_0^1\int_0^1 \frac{\lambda^4}{\Delta}\left(h^3\psi^{IV}_{nm,yyyy} + 6h^2h'_y\psi'''_{nm,yyy} + 3\left[2h(h'_y)^2 + h^2h''_{yy}\right]\psi''_{nm,yy} + 12k_y^2h\psi_{nm}\right)\psi_{kl}dxdy\,;$$

$$f_{8klnm} = \int_0^1\int_0^1 \frac{\mu_1\lambda^2}{\Delta}\left(h^3\psi^{IV}_{nm,xxyy} + 6h^2h'_y\psi'''_{nm,xxy} + 3\left[2h(h'_y)^2 + h^2h''_{yy}\right]\psi''_{nm,xx} + 12k_xk_yh\psi_{nm}\right)\psi_{kl}dxdy\,;$$

$$f_{9klnm} = \int_0^1\int_0^1 (1-\mu_1\mu_2)g\lambda^2\left(4h^3\psi^{IV}_{nm,xxyy} + 12h^2h'_x\psi'''_{nm,xyy} + 12h^2h'_y\psi'''_{nm,xxy} + \right.$$

$$\left. + 12\left[2hh'_xh'_y + h^2h''_{xy}\right]\psi''_{nm,xy}\right)\psi_{kl}dxdy\,;$$

$$\xi_{1klnmij} = 12\int_0^1\int_0^1 \lambda\delta\Delta\left(h\psi'_{nm,x}\phi''_{ij,xx} + h'_x\psi'_{nm,x}\phi'_{ij,x} + h\psi''_{nm,xx}\phi'_{ij,x}\right)\psi_{kl}dxdy\,;$$

$$\xi_{2klnmij} = 12\int_0^1\int_0^1 \frac{\mu_1\lambda^3\delta}{\Delta}\left(h\psi'_{nm,y}\phi''_{ij,xy} + h'_y\psi'_{nm,y}\phi'_{ij,x} + h\psi''_{nm,yy}\phi'_{ij,x}\right)\psi_{kl}dxdy\,;$$

$$\xi_{3klnmij} = 12\int_0^1\int_0^1 (1-\mu_1\mu_2)g\lambda^3\delta\left(h\psi'_{nm,x}\phi''_{ij,yy} + h'_y\psi'_{nm,y}\phi'_{ij,y} + h\psi'_{nm,y}\phi''_{ij,xy} + \right.$$

$$\left. + h'_x\psi'_{nm,y}\phi'_{ij,y} + 2h\psi''_{nm,xy}\phi'_{ij,y}\right)\psi_{kl}dxdy\,;$$

$$\xi_{4klnmij} = 12\int_0^1\int_0^1 \frac{\lambda^4\delta}{\Delta}\left(h\psi'_{nm,y}\varphi''_{ij,yy} + h'_y\psi'_{nm,y}\varphi'_{ij,y} + h\psi''_{nm,yy}\varphi'_{ij,y}\right)\psi_{kl}dxdy\,;$$

$$\xi_{5klnmij} = 12\int_0^1\int_0^1 \mu_2\lambda^2\delta\Delta\left(h\psi'_{nm,x}\varphi''_{ij,xy} + h'_x\psi'_{nm,x}\varphi'_{ij,y} + h\psi''_{nm,xx}\varphi'_{ij,y}\right)\psi_{kl}dxdy\,;$$

$$\xi_{6klnmij} = 12\int_0^1\int_0^1 (1-\mu_1\mu_2)g\lambda^2\delta\left(h\psi'_{nm,x}\varphi''_{ij,xy} + h'_y\psi'_{nm,x}\varphi'_{ij,x} + h\psi'_{nm,y}\varphi''_{ij,xx} + \right.$$

$$\left. + h'_x\psi'_{nm,y}\varphi'_{ij,x} + 2h\psi''_{nm,xy}\varphi'_{ij,x}\right)\psi_{kl}dxdy\,;$$

$$\xi_{7klnmij} = -12\int_0^1\int_0^1 k_x\Delta\left(h\psi'_{nm,x}\psi'_{ij,x} + h'_x\psi'_{nm,x}\psi_{ij} + h\psi''_{nm,xx}\psi_{ij}\right)\psi_{kl}dxdy\,;$$

$$\xi_{8klnmij}=-12\int_0^1\int_0^1 \lambda^2\mu_2 k_y\Delta\left(h\psi'_{nm,x}\psi'_{ij,x}+h'_x\psi'_{nm,x}\psi_{ij}+h\psi''_{nm,xx}\psi_{ij}\right)\psi_{kl}dxdy;$$

$$\xi_{9klnmij}=-12\int_0^1\int_0^1 \frac{\lambda^4 k_y}{\Delta}\left(h\psi'_{nm,y}\psi'_{ij,y}+h'_y\psi'_{nm,y}\psi_{ij}+h\psi''_{nm,yy}\psi_{ij}\right)\psi_{kl}dxdy;$$

$$\xi_{10klnmij}=-12\int_0^1\int_0^1 \frac{\lambda^2\mu_1 k_x}{\Delta}\left(h\psi'_{nm,y}\psi'_{ij,y}+h'_y\psi'_{nm,y}\psi_{ij}+h\psi''_{nm,yy}\psi_{ij}\right)\psi_{kl}dxdy;$$

$$g_{1klnmij}=6\int_0^1\int_0^1 k_x\Delta h\psi'_{nm,x}\psi'_{ij,x}\psi_{kl}dxdy;\ g_{2klnmij}=6\int_0^1\int_0^1 \lambda^2\mu_2 k_x\Delta h\psi'_{nm,y}\psi'_{ij,y}\psi_{kl}dxdy;$$

$$g_{3klnmij}=6\int_0^1\int_0^1 \frac{\lambda^2\mu_1 k_y}{\Delta}h\psi'_{nm,x}\psi'_{ij,x}\psi_{kl}dxdy;\ g_{4klnmij}=6\int_0^1\int_0^1 \frac{\lambda^4 k_y}{\Delta}h\psi'_{nm,y}\psi'_{ij,y}\psi_{kl}dxdy;$$

$$g_{5klnmijrs}=12\int_0^1\int_0^1 \Delta\left(h\psi'_{nm,x}\psi'_{ij,x}\psi''_{rs,xx}+\frac{1}{2}h'_x\psi'_{nm,x}\psi'_{ij,x}\psi'_{rs,x}+\frac{1}{2}h\psi''_{nm,xx}\psi'_{ij,x}\psi'_{rs,x}\right)\psi_{kl}dxdy;$$

$$g_{6klnmijrs}=12\int_0^1\int_0^1 \mu_2\Delta\lambda^2\left(h\psi'_{nm,x}\psi'_{ij,y}\psi''_{rs,xy}+\frac{1}{2}h'_x\psi'_{nm,x}\psi'_{ij,y}\psi'_{rs,y}+\frac{1}{2}h\psi''_{nm,xx}\psi'_{ij,y}\psi'_{rs,y}\right)\psi_{kl}dxdy;$$

$$g_{7klnmijrs}=12\int_0^1\int_0^1 \frac{\lambda^4}{\Delta}\left(h\psi'_{nm,y}\psi'_{ij,y}\psi''_{rs,yy}+\frac{1}{2}h'_y\psi'_{nm,y}\psi'_{ij,y}\psi'_{rs,y}+\frac{1}{2}h\psi''_{nm,yy}\psi'_{ij,y}\psi'_{rs,y}\right)\psi_{kl}dxdy;$$

$$g_{8klnmijrs}=12\int_0^1\int_0^1 \frac{\mu_1\lambda^2}{2\Delta}\left(2h\psi'_{nm,y}\psi'_{ij,x}\psi''_{rs,xy}+h'_y\psi'_{nm,y}\psi'_{ij,x}\psi'_{rs,x}+h\psi''_{nm,yy}\psi'_{ij,x}\psi'_{rs,x}\right)\psi_{kl}dxdy;$$

$$g_{9klnmijrs}=12\int_0^1\int_0^1 (1-\mu_1\mu_2)g\lambda^2\left(h\psi'_{nm,x}\psi'_{ij,y}\psi''_{rs,xy}+h\psi'_{nm,x}\psi'_{ij,x}\psi''_{rs,yy}+\right.$$

$$+h'_y\psi'_{nm,x}\psi'_{ij,x}\psi'_{rs,y}+h\psi'_{nm,y}\psi'_{ij,x}\psi''_{rs,xy}+h\psi'_{nm,y}\psi'_{ij,y}\psi''_{rs,xx}+$$

$$\left.+h'_x\psi'_{nm,y}\psi'_{ij,x}\psi'_{rs,y}+2h\psi''_{nm,xy}\psi'_{ij,x}\psi'_{rs,y}\right)\psi_{kl}dxdy;\ q_{kl}=q\int_0^1\int_0^1 \psi_{kl}dxdy;$$

$$p^2_{klnm}=f_{5klnm}+f_{6klnm}+f_{7klnm}+f_{8klnm}+f_{9klnm}-4\pi^2\lambda^2p^*_{klnm}\delta_0;\ \mu_{klnm}=\frac{2\pi^2\lambda^2p^*_{klnm}}{p^2_{klnm}}\delta_1.$$

System (14) was integrated using a numerical method based on the use of quadrature formulas [28]. Assuming harmonic oscillations, system (14) in integral form is obtained by integrating it twice over time *t*:

$$\sum_{n=1}^{N}\sum_{m=1}^{M}a_{klnm}u_{nm}=\sum_{n=1}^{N}\sum_{m=1}^{M}a_{klnm}\left(u_{0nm}+\dot{u}_{0nm}t\right)+\eta_1\int_0^t\int_0^\tau\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma^*_{11}\right)d_{1klnm}+\left(1-\Gamma^*\right)d_{2klnm}\right]u_{nm}+\right.\right.$$

$$\left.+\left[\left(1-\Gamma^*_{12}\right)d_{3klnm}+\left(1-\Gamma^*\right)d_{4klnm}\right]v_{nm}+\left[\left(1-\Gamma^*_{11}\right)d_{5klnm}+\left(1-\Gamma^*_{12}\right)d_{6klnm}\right]w_{nm}\right\}+$$

$$\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma^*_{11}\right)d_{7klnmij}+\left(1-\Gamma^*_{12}\right)d_{8klnmij}+\left(1-\Gamma^*\right)d_{9klnmij}\right]\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)\right\}d\tau ds,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}v_{nm} = \sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}\left(v_{0nm}+\dot{v}_{0nm}t\right)+\eta_2\int_0^t\int_0^\tau\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{21}^*\right)e_{1klnm}+\left(1-\Gamma^*\right)e_{2klnm}\right]u_{nm}+\right.\right.$$

$$+\left[\left(1-\Gamma_{22}^*\right)e_{3klnm}+\left(1-\Gamma^*\right)e_{4klnm}\right]v_{nm}+\left[\left(1-\Gamma_{21}^*\right)e_{5klnm}+\left(1-\Gamma_{22}^*\right)e_{6klnm}\right]w_{nm}\Big\}+ \quad (15)$$

$$\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma_{22}^*\right)e_{7klnmij}+\left(1-\Gamma_{21}^*\right)e_{8klnmij}+\left(1-\Gamma^*\right)e_{9klnmij}\right]\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)\right\}d\tau ds,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}w_{nm} = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{klnm}\left(w_{0nm}+\dot{w}_{0nm}t\right)-\eta_3\int_0^t\int_0^\tau\left\{\sum_{n=1}^{N}\sum_{m=1}^{M} p_{klnm}^2\left(1-2\mu_{klnm}\cos\Theta t\right)w_{nm}-\right.$$

$$-\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{11}^*\right)f_{1klnm}+\left(1-\Gamma_{21}^*\right)f_{2klnm}\right]u_{nm}+\left[\left(1-\Gamma_{12}^*\right)f_{3klnm}+\left(1-\Gamma_{22}^*\right)f_{4klnm}\right]v_{nm}+\right.$$

$$+\left[\Gamma_{11}^*f_{5klnm}+\Gamma_{12}^*f_{6klnm}+\Gamma_{22}^*f_{7klnm}+\Gamma_{21}^*f_{8klnm}+\Gamma^*f_{9klnm}\right]w_{0nm}\Big\}-$$

$$-\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} w_{nm}\left\{\left[\left(1-\Gamma_{11}^*\right)\xi_{1klnmij}+\left(1-\Gamma_{21}^*\right)\xi_{2klnmij}+\left(1-\Gamma^*\right)\xi_{3klnmij}\right]u_{ij}+\right.$$

$$+\left[\left(1-\Gamma_{22}^*\right)\xi_{4klnmij}+\left(1-\Gamma_{12}^*\right)\xi_{5klnmij}+\left(1-\Gamma^*\right)\xi_{6klnmij}\right]v_{ij}+$$

$$+\left[\left(1-\Gamma_{11}^*\right)\xi_{7klnmij}+\left(1-\Gamma_{12}^*\right)\xi_{8klnmij}+\left(1-\Gamma_{22}^*\right)\xi_{9klnmij}+\left(1-\Gamma_{21}^*\right)\xi_{10klnmij}\right]\left(w_{ij}-w_{0ij}\right)\Big\}+$$

$$+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left\{\left(1-\Gamma_{11}^*\right)g_{1klnmijrs}+\left(1-\Gamma_{12}^*\right)g_{2klnmijrs}+\left(1-\Gamma_{21}^*\right)g_{3klnmijrs}+\left(1-\Gamma_{22}^*\right)g_{4klnmijrs}\right\}\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)+$$

$$+\sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M} w_{nm}\left\{\left(1-\Gamma_{11}^*\right)g_{5klnmijrs}+\left(1-\Gamma_{12}^*\right)g_{6klnmijrs}+\left(1-\Gamma_{22}^*\right)g_{7klnmijrs}+\right.$$

$$\left.\left.+\left(1-\Gamma_{21}^*\right)g_{8klnmijrs}+\left(1-\Gamma^*\right)g_{9klnmijrs}\right\}\left(w_{ij}w_{rs}-w_{0ij}w_{0rs}\right)-12\eta_3\left(1-\mu_1\mu_2\right)\lambda^4q_{kl}\right\}d\tau ds.$$

By the formula for replacing the double integral with a single integral the system (15) is given in the following form:

$$\sum_{n=1}^{N}\sum_{m=1}^{M} a_{klnm}u_{nm} = \sum_{n=1}^{N}\sum_{m=1}^{M} a_{klnm}\left(u_{0nm}+\dot{u}_{0nm}t\right)+\eta_1\int_0^t(t-\tau)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{11}^*\right)d_{1klnm}+\left(1-\Gamma^*\right)d_{2klnm}\right]u_{nm}+\right.\right.$$

$$+\left[\left(1-\Gamma_{12}^*\right)d_{3klnm}+\left(1-\Gamma^*\right)d_{4klnm}\right]v_{nm}+\left[\left(1-\Gamma_{11}^*\right)d_{5klnm}+\left(1-\Gamma_{12}^*\right)d_{6klnm}\right]w_{nm}\Big\}+$$

$$\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma_{11}^*\right)d_{7klnmij}+\left(1-\Gamma_{12}^*\right)d_{8klnmij}+\left(1-\Gamma^*\right)d_{9klnmij}\right]\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)\right\}d\tau,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}v_{nm} = \sum_{n=1}^{N}\sum_{m=1}^{M} b_{klnm}\left(v_{0nm}+\dot{v}_{0nm}t\right)+\eta_2\int_0^t(t-\tau)\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\left[\left(1-\Gamma_{21}^*\right)e_{1klnm}+\left(1-\Gamma^*\right)e_{2klnm}\right]u_{nm}+\right.\right.$$

$$+\left[\left(1-\Gamma_{22}^*\right)e_{3klnm}+\left(1-\Gamma^*\right)e_{4klnm}\right]v_{nm}+\left[\left(1-\Gamma_{21}^*\right)e_{5klnm}+\left(1-\Gamma_{22}^*\right)e_{6klnm}\right]w_{nm}\Big\}+ \quad (16)$$

$$+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[\left(1-\Gamma_{22}^{*}\right)e_{7klnmij}+\left(1-\Gamma_{21}^{*}\right)e_{8klnmij}+\left(1-\Gamma^{*}\right)e_{9klnmij}\right]\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)\Bigg\}d\tau,$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M}c_{klnm}w_{nm}=\sum_{n=1}^{N}\sum_{m=1}^{M}c_{klnm}\left(w_{0nm}+\dot{w}_{0nm}t\right)-\eta_3\int_0^t(t-\tau)\Bigg\{\sum_{n=1}^{N}\sum_{m=1}^{M}p_{klnm}^2\left(1-2\mu_{klnm}\cos\Theta t\right)w_{nm}-$$

$$-\sum_{n=1}^{N}\sum_{m=1}^{M}\Big\{\left[\left(1-\Gamma_{11}^{*}\right)f_{1klnm}+\left(1-\Gamma_{21}^{*}\right)f_{2klnm}\right]u_{nm}+\left[\left(1-\Gamma_{12}^{*}\right)f_{3klnm}+\left(1-\Gamma_{22}^{*}\right)f_{4klnm}\right]v_{nm}+$$

$$+\left[\Gamma_{11}^{*}f_{5klnm}+\Gamma_{12}^{*}f_{6klnm}+\Gamma_{22}^{*}f_{7klnm}+\Gamma_{21}^{*}f_{8klnm}+\Gamma^{*}f_{9klnm}\right]w_{0nm}\Big\}-$$

$$-\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}w_{nm}\Big\{\left[\left(1-\Gamma_{11}^{*}\right)\xi_{1klnmij}+\left(1-\Gamma_{21}^{*}\right)\xi_{2klnmij}+\left(1-\Gamma^{*}\right)\xi_{3klnmij}\right]u_{ij}+$$

$$+\left[\left(1-\Gamma_{22}^{*}\right)\xi_{4klnmij}+\left(1-\Gamma_{12}^{*}\right)\xi_{5klnmij}+\left(1-\Gamma^{*}\right)\xi_{6klnmij}\right]v_{ij}+$$

$$+\left[\left(1-\Gamma_{11}^{*}\right)\xi_{7klnmij}+\left(1-\Gamma_{12}^{*}\right)\xi_{8klnmij}+\left(1-\Gamma_{22}^{*}\right)\xi_{9klnmij}+\left(1-\Gamma_{21}^{*}\right)\xi_{10klnmij}\right]\left(w_{ij}-w_{0ij}\right)\Big\}+$$

$$+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\Big\{\left(1-\Gamma_{11}^{*}\right)g_{1klnmijrs}+\left(1-\Gamma_{12}^{*}\right)g_{2klnmijrs}+\left(1-\Gamma_{21}^{*}\right)g_{3klnmijrs}+\left(1-\Gamma_{22}^{*}\right)g_{4klnmijrs}\Big\}\left(w_{nm}w_{ij}-w_{0nm}w_{0ij}\right)+$$

$$+\sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M}w_{nm}\Big\{\left(1-\Gamma_{11}^{*}\right)g_{5klnmijrs}+\left(1-\Gamma_{12}^{*}\right)g_{6klnmijrs}+\left(1-\Gamma_{22}^{*}\right)g_{7klnmijrs}+$$

$$+\left(1-\Gamma_{21}^{*}\right)g_{8klnmijrs}+\left(1-\Gamma^{*}\right)g_{9klnmijrs}\Big\}\left(w_{ij}w_{rs}-w_{0ij}w_{0rs}\right)-12\eta_3\left(1-\mu_1\mu_2\right)\lambda^4q_{kl}\Bigg\}d\tau,$$

$$u_{nm}(0)=u_{0nm},\ \dot{u}_{nm}(0)=\dot{u}_{0nm},\ v_{nm}(0)=v_{0nm},\ \dot{v}_{nm}(0)=\dot{v}_{0nm},$$

$$w_{nm}(0)=w_{0nm},\ \dot{w}_{nm}(0)=\dot{w}_{0nm},\ k=1,2,...,N;\ l=1,2,...,M.$$

Assuming that $t=t_i$, $t_i=i\Delta t, i=1,2,...$ (where Δt is the integration step) and replacing the integrals with quadrature trapezoidal formulas to calculate the unknowns $w_{inm}=w_{inm}(t_i)$, $u_{inm}=u_{inm}(t_i)$ and $v_{inm}=v_{inm}(t_i)$, a system of recurrent formulas is obtained.

Based on the developed algorithm, a program was compiled in the Delphi algorithmic language.

3. Results and Discussion

The results of calculations for various physical and geometric parameters are shown in graphs in Fig. 3–7. Numerical results are compared to the ones available in the literature.

The effect of orthotropic properties of the material on the behavior of a shell was studied (Fig. 3). As seen from the figure, an increase in parameter Δ that determines the degree of anisotropy (curve 1 - Δ=1; curve 2 - Δ=1.5; curve 3 - Δ=2.0) leads to an increase in the oscillation amplitude and a phase shift to the left.

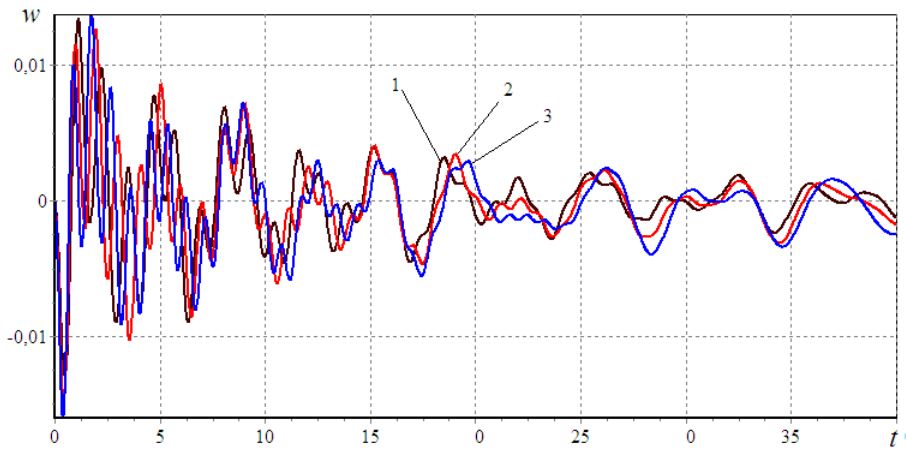


Figure 3. Dependence of the deflection vs time for

$$\lambda = 1; \delta = 25; k_x = 10; k_y = 10; q = 0; p_x = 0; p_y = 0; \alpha^* = 0.5; \Theta = 1.1;$$

$$A = A_{ij} = 0.05, i, j = 1,2;$$

Δ =1 (1); 1.5 (2); 2.0 (2)

Figure 4 shows the results obtained from different theories. Here, curve 1 corresponds to the case when the shell material is elastic, curve 2 - to the case when the viscosity of the material is taken into account in the direction of shear $\left(A = 0.05, A_{ij} = 0, i, j = 1,2\right)$, and curve 3 - to the case when the viscosity is taken into account in all directions $\left(A = A_{ij} = 0.05, i, j = 1,2\right)$.

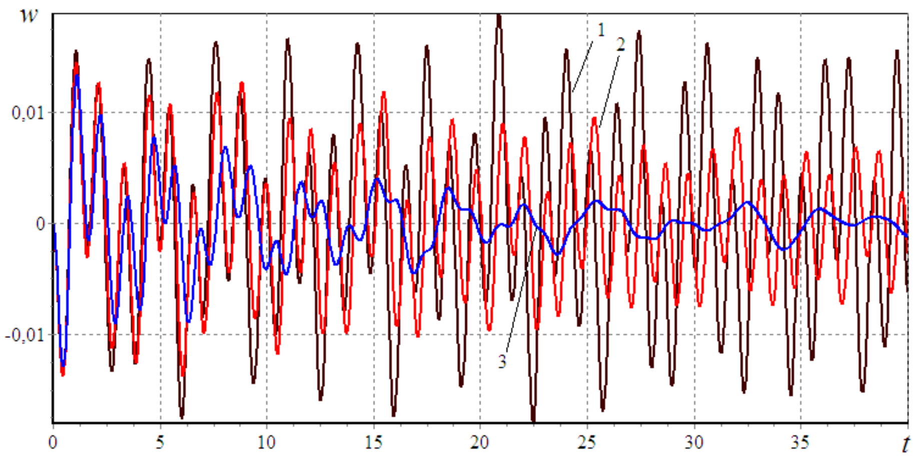


Figure 4. Dependence of the deflection vs time for

$$\lambda = 1; \ \delta = 25; \ k_x = 10; \ k_y = 10; \ q = 0; \ p_x = 0; \ p_y = 0; \ \alpha^* = 0.5; \ \Theta = 1.1; \ \Delta = 1$$

The results obtained confirm that viscous-elastic properties of the material should be considered not only in the shear direction but also in other directions.

The influence of the shell thickness on its behavior is studied. Figure 5 shows the results obtained for various values of the thickness change parameter α^*. It can be seen that with an increase in this parameter, the oscillation amplitude increases. In particular, the results obtained for a shallow shell of constant thickness ($\alpha^* = 0$) coincide with the results obtained in [29].

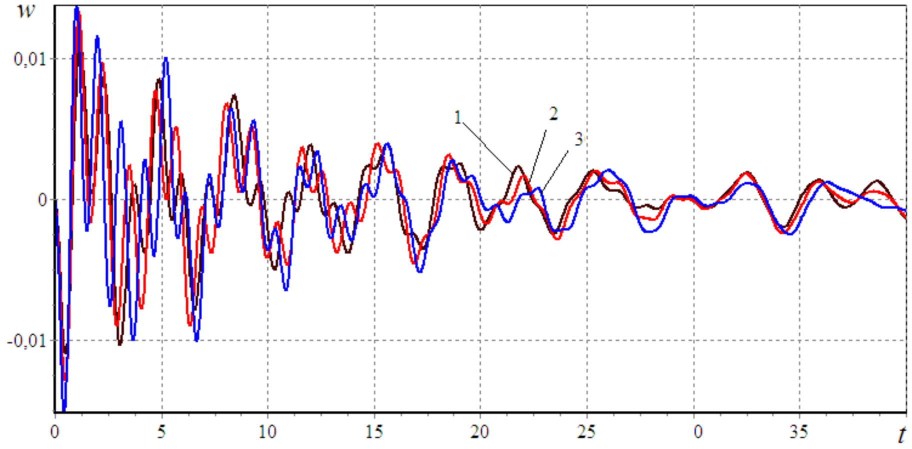


Fig.5. Dependence of the deflection vs time for

$$\lambda = 1; \delta = 25; k_x = 10; k_y = 10; q = 0; p_x = 0; p_y = 0; \Theta = 1.1;$$

$$A = A_{ij} = 0.05, i, j = 1,2; \Delta = 1$$

α^*=0 (1); 0.5 (2); 0.8 (3)

Figure 6 shows the results obtained for various values of the curvature parameter k_x. An increase in this parameter leads to an increase in the amplitude of oscillations.

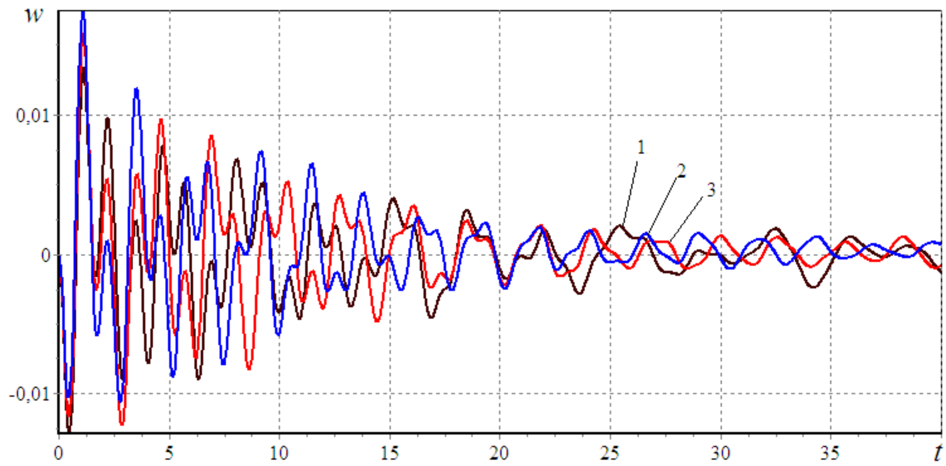


Figure 6. Dependence of the deflection vs time for

$$\lambda = 1; \delta = 25; k_y = 10; q = 0; p_x = 0; p_y = 0; \alpha^* = 0.5; \Theta = 1.1;$$

$$A = A_{ij} = 0.05, i, j = 1,2; \Delta = 1$$

k_x=10 (1); 15 (2); 20 (3)

Figure 7 shows the results obtained for various values of the frequency of the external periodic load Θ. An increase in this parameter leads to an increase in the amplitude of oscillations.

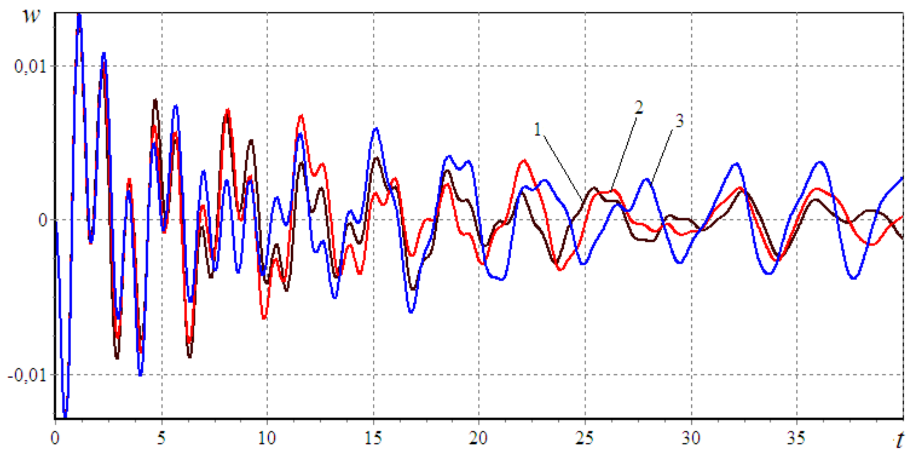


Figure 7. Dependence of the deflection vs time for

$$\lambda = 1;\ \delta = 25;\ k_x = 10;\ k_y = 10;\ q = 0;\ p_x = 0;\ p_y = 0;\ \alpha^* = 0.5;$$

$$A = A_{ij} = 0.05,\ i,\ j = 1,2;\ \Delta = 1$$

Θ =1.1 (1); 1.3 (2); 1.5 (3)

4. Conclusion

A mathematical model, method, and computer program were developed to estimate parametric oscillations of a viscous-elastic orthotropic shallow shell of variable thickness, taking into account geometric nonlinearity under periodic loads.

The dynamic stability of a viscous-elastic orthotropic shallow shell of variable thickness was described by a nonlinear system of IDEs.

The application of Galerkin method with the discretization of spatial variables at each time point reduces the problem of dynamic stability of a viscous-elastic orthotropic shallow shell of variable thickness to solving a non-decaying system of ordinary nonlinear IDEs with weakly singular kernels with variable coefficients.

The impact on the amplitude-time characteristics and the SSS of a change in the physical-mechanical and geometric parameters of the shell material was estimated.

The method proposed in this article can be used for various types of thin-wall structures (plates, panels, and shells of variable thickness).

References

1. Bolotin, V. V. Dynamic Stability of Structures. Nonlinear Stability of Structures: Theory and Computational Techniques. Vienna, Springer Vienna, 1995. Pp. 3–72.
2. Alijani, F., Amabili, M. Non-linear vibrations of shells: A literature review from 2003 to 2013. International Journal of Non-Linear Mechanics. 2014. 58. Pp. 233–257. DOI: 10.1016/j.ijnonlinmec.2013.09.012
3. Amabili, M. Nonlinear vibrations and stability of shells and plates. New York, USA, Cambridge University Press, 2008. 374 p. ISBN:978-0521883290.
4. Radwańska, M., Stankiewicz, A., Wosatko, A. Plate and Shell Structures: Selected Analytical and Finite Element Solutions. John Wiley & Sons, 2017. 424 p. ISBN:9781118934548.
5. H., S.A., Zihni, Z., P., A.B., David, H., Ferruh, T., Hakan, E. The dynamic instability of FG orthotropic conical shells within the SDT. Steel and Composite Structures. 2017. 25(5). Pp. 581–591. DOI: 10.12989/SCS.2017.25.5.581
6. Talimian, A., Beda, P. Dynamic Stability of a Thin Plate Subjected to Bi-Axial Edged Loads. Acta Polytechnica Hungarica. 2018. 15(2). Pp. 125–139. DOI: 10.12700/aph.15.1.2018.2.7
7. Kosheleva, E. Plate stability under periodic load action, material of which obeys hereditary law of viscoelasticity. Proceedings of the 2018 7th International Conference on Energy, Environment and Sustainable Development (ICEESD 2018). 2018. Pp. 568–572. DOI: 10.2991/iceesd-18.2018.103
8. Sofiyev, A.H. Influences of shear deformations and material gradient on the linear parametric instability of laminated orthotropic conical shells. Composite Structures. 2019. 225. Pp. 111156. DOI: 10.1016/j.compstruct.2019.111156
9. Dey, T., Jansen, E., Kumar, R., Rolfes, R. Instability characteristics of variable stiffness laminated composite curved panels under non-uniform periodic excitation. Thin-Walled Structures. 2022. 171. Pp. 108735. DOI: 10.1016/j.tws.2021.108735
10. Becerra, H.C., Silva, A.R. Parametric study using a curved shell finite element to dynamic analysis of footbridge under rhythmic loading. Structures. 2021. 34. Pp. 135–148. DOI: 10.1016/j.istruc.2021.07.057
11. Bakulin, V.N., Nedbai, A.Y. Dynamic Stability of a Composite Cylindrical Shell with Linear-Variable Thickness under Pulsed External Pressure. Journal of Engineering Physics and Thermophysics. 2021. 94(2). Pp. 525–533. DOI: 10.1007/s10891-021-02324-7

12. Shahmohammadi, M.A., Mirfatah, S.M., Salehipour, H., Azhari, F., Civalek, Ö. Dynamic stability of hybrid fiber/nanocomposite-reinforced toroidal shells subjected to the periodic axial and pressure loadings. Mechanics of Advanced Materials and Structures. 2022. Pp. 1–17. DOI: 10.1080/15376494.2022.2037172
13. Phu, K. Van, Bich, D.H., Doan, L.X. Nonlinear Dynamic Stability of Variable Thickness FGM Cylindrical Shells Subjected to Mechanical Load. Modern Mechanics and Applications. 2022. Pp. 506–521.
14. Sieber, J., Hutchinson, J.W., Thompson, J.M.T. Nonlinear dynamics of spherical shells buckling under step pressure. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 2019. 475(2223). Pp. 20180884. DOI:10.1098/rspa.2018.0884
15. Krysko, A. V, Awrejcewicz, J., Mitskevich, S.A., Zhigalov, M. V, Krysko, V.A. Nonlinear dynamics of heterogeneous shells. Part 2. Chaotic dynamics of variable thickness shells. International Journal of Non-Linear Mechanics. 2021. 129. Pp. 103660. DOI: 10.1016/j.ijnonlinmec.2020.103660
16. Fu, T., Wu, X., Xiao, Z., Chen, Z. Study on dynamic instability characteristics of functionally graded material sandwich conical shells with arbitrary boundary conditions. Mechanical Systems and Signal Processing. 2021. 151. Pp. 107438. DOI: 10.1016/j.ymssp.2020.107438
17. Safaei, B. Frequency-dependent damped vibrations of multifunctional foam plates sandwiched and integrated by composite faces. The European Physical Journal Plus. 2021. 136(6). Pp. 646. DOI: 10.1140/epjp/s13360-021-01632-4
18. Zaczynska, M., Abramovich, H., Bisagni, C. Parametric studies on the dynamic buckling phenomenon of a composite cylindrical shell under impulsive axial compression. Journal of Sound and Vibration. 2020. 482. Pp. 115462. DOI: 10.1016/j.jsv.2020.115462
19. Chen, C.-S., Wang, H., Yeh, C.-C., Chen, W.-R. Dynamic Instability Response of Soft Core Sandwich Plates Based on Higher-Order Plate Theory. International Journal of Structural Stability and Dynamics. 2021. 21(09). Pp. 2150118. DOI: 10.1142/S0219455421501182
20. Lin, G., Li, F., Zhang, Q., Chen, P., Sun, W., Saikov, I., Shcherbakov, V., Alymov, M. Dynamic instability of fiber composite cylindrical shell with metal liner subjected to internal pulse loading. Composite Structures. 2022. 280. Pp. 114906. DOI: 10.1016/j.compstruct.2021.114906
21. Ma, L., Liu, X., Moradi, Z. On the chaotic behavior of graphene-reinforced annular systems under harmonic excitation. Engineering with Computers. 2022. 38(3). Pp. 2583–2607. DOI: 10.1007/s00366-020-01210-9
22. Normuminov, B., Abdikarimov, R., Mirsaidov, M., Khodzhaev, D., Mirzaev, B. Parametric vibrations of viscoelastic orthotropic cylindrical panels of variable thickness. {IOP} Conference Series: Materials Science and Engineering. 2020. 869. Pp. 52034. DOI: 10.1088/1757-899x/869/5/052034
23. Khodzhaev, D.A., Abdikarimov, R.A., Mirsaidov, M.M. Dynamics of a physically nonlinear viscoelastic cylindrical shell with a concentrated mass. Magazine of Civil Engineering. 2019. 91(7). Pp. 39–48. DOI: 10.18720/MCE.91.4
24. Abdikarimov, R., Amabili, M., Vatin, N.I., Khodzhaev, D. Dynamic Stability of Orthotropic Viscoelastic Rectangular Plate of an Arbitrarily Varying Thickness. Applied Sciences. 2021. 11(13). DOI: 10.3390/app11136029
25. Reddy, J.N. Theory and analysis of elastic plates and shells. 2nd Editio . CRC Press, 2007. 547 p. ISBN:9780849384158.
26. Ambartsumyan, S.A. Theory Of Anisotropic Plates: Strength, Stability, & Vibrations. CRC Press, 1991. 360 p.
27. Mal'tsev, L.E. The analytical determination of the Rzhanitsyn-Koltunov nucleus. Mechanics of Composite Materials. 1979. 15(1). Pp. 131–133.
28. Abdikarimov, R.A., Khodzhaev, D.A. Computer modeling of tasks in dynamics of viscoelastic thinwalled elements in structures of variable thickness. Magazine of Civil Engineering. 2014. 49(5). Pp. 83–94. DOI: 10.5862/MCE.49.9
29. Harvan, J., Psorty, M. Stability analysis of a shallow shell. Procedia Engineering. 2017. 190. Pp. 148–153. DOI: 10.1016/j.proeng.2017.05.320

Information about authors:

Dadakhan Khodzhaev, *PhD in Physics and Mathematics*
ORCID: https://orcid.org/0000-0001-5526-8723
E-mail: dhodjaev@mail.ru

Rustamkhan Abdikarimov*, Doctor of Physics and Mathematics*
ORCID: https://orcid.org/0000-0001-8114-1187
E-mail: rabdikarimov@mail.ru

Marco Amabili, *PhD*
ORCID: https://orcid.org/0000-0001-9340-4474
E-mail: marco.amabili@mcgill.ca

Bakhodir Normuminov,
ORCID: https://orcid.org/0000-0002-5018-3533
E-mail: normuminovb_77@inbox.ru

Received 11.10.2022. Approved after reviewing 20.03.2023. Accepted 23.03.2023.