Magazine of Civil Engineering. 2024. 17(1). Article No. 12508

Magazine of Civil Engineering

journal homepage: http://engstroy.spbstu.ru/

ISSN 2712-8172

Research article
UDC 626.31
DOI: 10.34910/MCE.125.8

Determining parameters of high-velocity open water flow

O.A. Burtseva[1] , S.I. Evtushenko[2] , V.N. Kokhanenko[1]

[1] Platov South-Russian State Polytechnic University (NPI), Novocherkassk, Russian Federation

[2] Moscow State University of Civil Engineering (National Research University), Moscow, Russian Federation

evtushenkosi@mgsu.ru

Keywords: mathematical model, 2D water flow, equations of motion, resistance forces, open-channel hydraulics, analytical solution, simple wave, characteristics method

Abstract. In the planar hydraulics, the problems with previously unknown boundaries are most difficult. The best adequacy in terms of flow parameters is provided by simplified analytical methods based on a potential flow model. A mathematical model of a stationary, potential, 2D planar high-velocity open water flow of an ideal fluid, freely spreading back from a non-pressure orifice is studied. The boundary problem of flow free spreading in plane were formulated. Studying the system of dimensionless equations of motion resulted in identification of the criteria influencing the process of flow spreading. A critical analysis was carried out and a description of various methods for solving the problem of free spreading of a high-velocity water flow was given. The problem in an analytical form was solved in the velocity hodograph plane. All flow parameters are determined in the physical plane. For the first time, the conjugating flow "simple wave" was applied. The proposed analytical method for solving the problem of flow free spreading is effective, unambiguous and has no singularities and discontinuities, particularly, at the outlet of a non-pressure pipe. The adequacy of the mathematical model was verified on a test example. The relative error of the flow parameters does not exceed 10 % compared to the experimental data.

Citation: Burtseva, O.A., Evtushenko, S.I., Kokhanenko, V.N. Determining parameters of high-velocity open water flow. Magazine of Civil Engineering. 2024. 17(1). Article no. 12508. DOI: 10.34910/MCE.125.8

1. *Introduction*

In modern hydraulic engineering construction, the hydraulic structures for passing water from elevated areas to lower ones are frequently used. These can be large hydro power plants, road drainage channels, spillways, junctions of various channels for changing flow parameters, small bridges that pass water flows during river floods or high water.

Due to inaccurate or irrational modeling in the construction of hydraulic structures, the operational reliability of a structure as a whole decreases; this causes collapses of fastenings of culverts under roads, small bridges under railways. Improper design of spillways leads to environmental disasters in Russia. There is a practical need for a reliable method for calculating the parameters of a water flow.

It was revealed that the methods of I.A. Sherenkov and G.A. Lilitsky, referred to as the most famous, accessible and described in the reference literature ones, do not always give results with sufficient accuracy for practical calculations when we compared the experimental and calculated contours of the border streamlines of flow free spreading downstream of rectangular cross-section pipes according to these methods.

Among the many tasks in the hydraulics of planar flows, the tasks with previously unknown flow boundaries are the most difficult. At the outlet of the flow from the pipe, the best adequacy in terms of flow parameters is represented by simplified analytical methods based on the model of the potential flow of the current. Further, the flow resistance forces increase and it is necessary to make a transition to numerical methods.

However, the fastening of the structure is carried out precisely at the culvert outlet for a flow, where an analytical solution may be enough. If the boundary value problem is immediately solved by numerical methods, then due to the characteristics of the problem (discontinuity of parameters in the pipe outlet area), the adequacy of the solution of the problem decreases. Therefore, first of all, it is necessary to use analytical methods as a basis for further use of numerical methods.

Therefore, there is a need for a simple analytical method for calculating the parameters of the water flow, which allows obtaining sufficient adequacy in terms of its parameters. Next, we consider a stationary, potential, high-velocity 2D planar open water flow of an ideal fluid freely spreading downstream of non-pressure culverts (hereinafter referred to as the water flow). Such a flow is characterized by local averaged velocities in depth and local depths at each point in the flow. The presence of velocities perpendicular to the planar flow plan distinguishes 3D flows from 2D planar flows. 2D water flows are studied and analyzed mathematically much easier than 3D spatial flows.

N. Bernadsky and V. Makkaveev were the first to set the problem of planar hydraulics. They also developed an approximate solving method for calm (precritical) flows. The theory of 2D planar flows was further developed in the works of Russian researchers N. Meleshchenko, G. Sukhomel, I. Levi, S. Numerov, F. Frankl, I. Sherenkov, B. Emtsev, A. Tursunov, N. Kartvelishvili, M. Mikhalev, V. Lyakhter, L. Vysotsky and foreign scientists A. Ippen, H. Rauz, D. Harleman, D. Dowson, R. Knapp, D. Liggett, T. Akatai, etc. The theory and methods for solving planar hydraulic problems are most fully described in monographs of G. Sukhomel, I. Levi, B. Yemtsev [1], I. Sherenkov [2], A. Yesin [3] and V. Kokhanenko [4].

There are two main types of engineering problems in the 2D planar hydrodynamics [2]. The first type, or the direct problems, includes the configuration of the planar channel, i.e. the shape of its banks, in addition to the bottom surface (topography). To supplement these data with known values of velocities and depths in one of the boundary cross-sections, it is necessary to find the form of the free surface and velocity distribution within the selected section of the channel. The second type of problems is the inverse problems, where the law of estimating certain hydraulic parameters is given and other flow parameters are found as well as geometric characteristics of the bed (relief of bottom) forming the indicated flow. At the same time, it is necessary to take into account the flow’s dynamic features and properties. The authors of this paper consider the second type of problems.

The mathematical basis of the problems mentioned above are systems of quasi-linear partial differential equations. The analytical solution is difficult to obtain in most cases due to the complexity of the motion. Therefore, a numerically analytical approach to solving such problems is considered promising.

Furthermore, the use of approximate methods of solution is recommended. The most famous approximate solution method is the method of characteristics developed by S. Chaplygin and borrowed from gas dynamics [4].

Sherenkov's method based on using a universal graph [2] constructed by means of the characteristics method is considered to be one of the most known approaches to determining parameters of a high-velocity flow. However, this method was not adequate enough for practical use. The discrepancy between the calculated and experimental values reached 50 % [2]. The characteristic method was further developed in works of V. Kokhanenko and his students.

In 1997, V. Kokhanenko [4] proposed the idea of an analytical method for determining flow parameters using the velocity hodograph plane, where such natural coordinates were proposed: τ – flow kineticity (the square of the flow velocity coefficient) and θ – angle of direction of the velocity vector in relation to symmetry axis OX of the flow. As the unknown functions we considered the potential function $\varphi = \varphi(\tau, \theta)$ and the current function $\psi = \psi(\tau, \theta)$. The basic system of differential equations of motion for the flow in the plane of the variables τ, θ has become linear, allowing a wide set of solutions to be defined.

E. Duvanskaya [5] calculated hydraulic parameters considering the friction forces and the slope of the bottom of 2D stationary precritical flows for solving problems at designing melioration networks and road structures.

This idea was further developed in the works of N. Kosichenko [6]. She set the problem about free flow spreading, solved it and indicated the limits of applicability of the results of solving of the planar problem. She proposed an analytical method for calculating the geometry of the flow spreading area and its parameters inside and at the boundary of the flow area.

N. Papchenko has developed new mathematical methods for modeling 2D planar flows. He determined a set of previously unknown analytical solutions for systems of 2D planar flows in the plane of the velocity hodograph. He founded solutions of the boundary problem of a free flow spreading both in the plane of velocity hodograph, and in the physical plane of the flow by both analytical and numerical methods [7].

The main contribution of D. Kelekhsaev to theory of planar flows was the definition of the inertial front length formula at the pipe outlet [8].

Substantial development of an analytical method for determining flow parameters was presented in the works of O. Burtseva and M. Aleksandrova [9–10]. They have formulated a boundary problem of free spreading of a planar flow. The system of equations of motion of water flow in dimensionless form was obtained in two ways. Criteria influencing the process of flow spreading were revealed and their universality was proved. The problem has been solved in the analytical form in the velocity hodograph plane. The transition to the physical plane made it possible to determine all parameters of plane water flow.

Improvement of the accuracy of the analytical method by splitting the flow scheme into four main sections: uniform flow, “simple wave”, section limited by the characteristics of the 1st family and the radial flow section. Conjugation the sections and obtaining an adequate mathematical model for determining the parameters of a high-velocity open water flow.

2. Methods

2.1. Motion equations of a 2D planar water flow in the physical region of a flow (OXY)

The initial physical assumptions for the 2D planar water flow model are [1]:

a) vertical components of local averaged velocities and accelerations are small;

b) velocity vectors of liquid particles located on the one vertical line are located in the same plane;

c) velocity distribution on any vertical is almost uniform.

In practice, in most open water flows the assumptions a), b), c) are satisfied by the high-velocity flows, the transverse dimensions of which are several times greater than the flow depth and there are no return currents [11–13].

Background of the paragraph c). The uneven distribution of open flow velocities along the vertical is conditioned by the retarding effect of the channel bottom. Herewith, a boundary layer develops which spreads its influence up to the free surface. However, the velocity distribution in the boundary layer has an asymptotic character and its conditional thickness is decreased with an increase in the flow kineticity. Therefore, in high-velocity flows (Froude number F > 1), the longitudinal component of the velocity at some point becomes almost equal to the surface one at a small distance from the bottom. Hence, there appears an almost uniform distribution of velocities along the vertical [18].

To obtain the dynamic equations of motion of a 2D open planar flow N.M. Bernadsky [1] suggested to proceed from L. Euler's equations, supplemented by components considering the flow resistance forces. For the flows satisfying the assumptions (a-c), the form of the motion equations for the average values of parameters in terms of depth and time coincides with the L. Euler’s equations for ideal liquid supplemented by the flow resistance force per mass unit of the liquid. This model has been obtained artificially but it has the right to both theoretical and practical use.

The system of dynamic equations of motion of a non-stationary open 2D planar flow has the form [1, 3]:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} + g \dfrac{\partial}{\partial x}\left(z_g + h\right) + T_x = 0; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} + g \dfrac{\partial}{\partial y}\left(z_g + h\right) + T_y = 0, \end{cases} \tag{1}$$

where u_x, u_y are projections of the local velocity vector on the axes of a Cartesian coordinate system – OXY; z_g is the mark of the channel bottom; h is flow depth; g is acceleration of gravity; T_x, T_y are the projections of resistance forces to the flow, referred to the unit mass of the liquid.

In the case of a flat horizontal channel bottom $z_g = 0,$ without taking into account the flow resistance forces $T_x = T_y = 0,$ from (1) there follows the system of equations:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} + g \dfrac{\partial}{\partial x}(h) = 0; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} + g \dfrac{\partial}{\partial y}(h) = 0. \end{cases} \tag{2}$$

Adding to the system (2) the flow continuity equation, which has a specific form (3) for 2D open water planar flows [1]:

$$\frac{\partial(u_x h)}{\partial x} + \frac{\partial(u_y h)}{\partial y} = 0 \tag{3}$$

and the flow potentiality equation

$$\Omega = \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} = 0 \tag{4}$$

we obtain a closed system of equations relatively to variables $u_x,\ u_y,\ h.$

2.2. Boundary value problem of flow free spreading in plane

The system of equations describing the flow is as follows:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} + g \dfrac{\partial h}{\partial x} = 0; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} + g \dfrac{\partial h}{\partial y} = 0; \\ \dfrac{\partial(u_x h)}{\partial x} + \dfrac{\partial(u_y h)}{\partial y} = 0, \\ \dfrac{\partial u_x}{\partial y} - \dfrac{\partial u_y}{\partial x} = 0. \end{cases} \tag{5}$$

One can go from $u_x,\ u_y$ to the local velocity vector V with modulus $V = \sqrt{u_x^2 + u_y^2}$ the angle θ characterizing the direction of the velocity vector to the longitudinal axis of flow symmetry [4, 18].

The boundary conditions for a flow:

1) at the outlet of a non-pressure rectangular pipe:

$$x = 0;\ \ -\frac{b}{2} \le y \le \frac{b}{2};\ \ h = h_0;\ \ V = V_0;\ \ \theta = 0, \tag{6}$$

where b is the width of the culvert; $h_0,\ V_0$ are the depth of the flow and rate of its velocity at the outlet of the pipe into a diversion channel;

2) along the boundary streamline

$$y = f(x);\ \ y'_x = tg\alpha;\ \ \alpha = 0, \tag{7}$$

α is wave angle;

3) at

$$x \to \infty;\ h \to 0;\ V \to V_{\max};\ \theta \to \theta_{\max}, \tag{8}$$

where V_{max}, θ_{max} are maximum flow velocity and spreading angle.

2.3. Research of the boundary value problem without its complete solution

Reducing the system of motion equations of 2D planar water flow and boundary conditions to dimensionless form.

Obtaining a system of motion equations of a water flow in a dimensionless form is necessary to identify dimensionless criteria that influence the process of flow spreading, and to answer the question, whether the type of the system and the boundary value problem are universal in dimensionless coordinates as a whole.

As a basis, we take the system of equations (5). Let us consider various variants for bringing it to a dimensionless form.

Variant I. We introduce the following dimensionless coordinates and dimensionless flow parameters

$$\overline{x} = \frac{x}{b};\ \overline{y} = \frac{y}{b};\ \overline{h} = \frac{h}{h_0};\ \overline{V}_{\overline{x}} = \frac{u_x}{V_0};\ \overline{V}_{\overline{y}} = \frac{u_y}{V_0},$$

where x, y are coordinates of the flow.

After transition in the system (5) to the dimensionless quantities $\overline{x},\ \overline{y},\ \overline{h},\ \overline{V}_{\overline{x}},\ \overline{V}_{\overline{y}}$, we obtain the following system:

$$\begin{cases} F_0\left(\overline{V}_{\overline{x}} \cdot \dfrac{\partial \overline{V}_{\overline{x}}}{\partial \overline{x}} + \overline{V}_{\overline{y}} \cdot \dfrac{\partial \overline{V}_{\overline{x}}}{\partial \overline{y}}\right) + \dfrac{\partial \overline{h}}{\partial \overline{x}} = 0; \\ F_0\left(\overline{V}_{\overline{x}} \cdot \dfrac{\partial \overline{V}_{\overline{y}}}{\partial \overline{x}} + \overline{V}_{\overline{y}} \cdot \dfrac{\partial \overline{V}_{\overline{y}}}{\partial \overline{y}}\right) + \dfrac{\partial \overline{h}}{\partial \overline{y}} = 0; \\ \dfrac{\partial\left(\overline{V}_{\overline{x}} \cdot \overline{h}\right)}{\partial \overline{x}} + \dfrac{\partial\left(\overline{V}_{\overline{y}} \cdot \overline{h}\right)}{\partial \overline{y}} = 0; \\ \dfrac{\partial \overline{V}_{\overline{x}}}{\partial \overline{y}} - \dfrac{\partial \overline{V}_{\overline{y}}}{\partial \overline{x}} = 0, \end{cases} \tag{9}$$

where $F_0 = \dfrac{V_0^2}{gh_0}$ is Froude criterion of a flow at the outlet from of a pipe.

Analyzing the system (9), we see that the first and the second equations are not reduced to a universal form as they contain the dimensionless parameter F_0. The third and the fourth equations have a universal form. Therefore, without solving the boundary value problem, we can say that its solution will depend on the Froude criterion F_0 of a flow at the outlet of a pipe and is not reduced to a universal form[1].

In case of the boundary value problem is universal, it can be solved one time. For the different boundary conditions, we can use only recalculation of the parameters describing the problem in dimensionless parameters.

Variant II. Further we show that from I.A. Sherenkov's graph [2] in coordinates

$$\overline{y} = \frac{y}{b};\quad \overline{x} = \frac{x}{b\sqrt{F_0}},$$

for the system (5) that he used, and for the boundary conditions (6), there follows a transformation:

---

1 A universal form is a form of the boundary value problem in the dimensionless coordinates, which does not depend on any dimensionless criteria.

$$\overline{y} = \frac{y}{b}; \quad \overline{x} = \frac{x}{b\sqrt{F_{r_0}}}; \quad \overline{h} = \frac{h}{h_0}; \quad V_{\overline{x}} = \frac{V_x}{V_0 K_x}; \quad V_{\overline{y}} = \frac{V_y}{V_0 K_y}.$$

The system of equations (5) is not reduced to the universal dimensionless form in this case: the first three equations at $K_x = \frac{1}{\sqrt{F_{r_0}}}; \quad K_y = \frac{1}{F_0}$ are reduced to a universal form, except the fourth one.

$$\begin{cases} V_{\overline{x}} \cdot \frac{\partial V_{\overline{x}}}{\partial \overline{x}} + V_{\overline{y}} \cdot \frac{\partial V_{\overline{x}}}{\partial \overline{y}} + \frac{\partial \overline{h}}{\partial \overline{x}} = 0; \\ V_{\overline{x}} \cdot \frac{\partial V_{\overline{y}}}{\partial \overline{x}} + V_{\overline{y}} \cdot \frac{\partial V_{\overline{y}}}{\partial \overline{y}} + \frac{\partial \overline{h}}{\partial \overline{y}} = 0; \\ \frac{\partial \left(V_{\overline{x}} \cdot \overline{h}\right)}{\partial \overline{x}} + \frac{\partial \left(V_{\overline{y}} \cdot \overline{h}\right)}{\partial \overline{y}} = 0; \\ \frac{\partial V_{\overline{x}}}{\partial \overline{y}} - \frac{1}{F_0} \frac{\partial V_{\overline{y}}}{\partial \overline{x}} = 0. \end{cases}$$

The boundary values (6-8) cannot be reduced to universal dimensionless form at the outlet of the pipe (6). That means that the problem of free spreading of the 2D flow cannot be reduced to the dimensionless form. Therefore, the universal graphic can be used only with the Froude criterion values close to 1 (see Fig. 1).

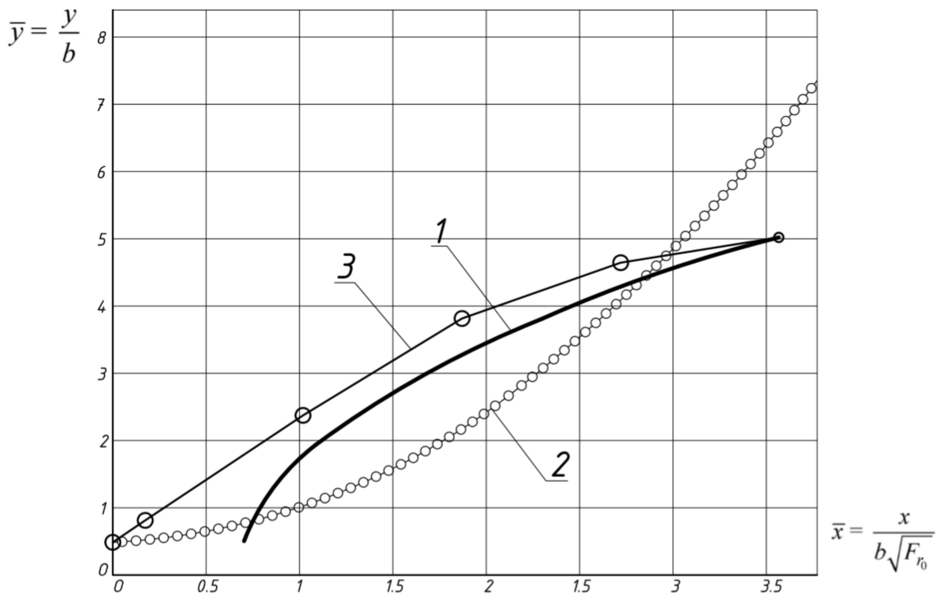


Figure 1. Graphs of boundary streamlines: 1 – according to the method of G.A. Lilitsky; 2 – according to the method of I.A. Sherenkov; 3 – according to the experimental data at $F_0 = 2.184$.

2.3.1. Methods for solving problem of free spreading of a water flow

I.A. Sherenkov's method. He developed a method based on the use of universal graphic with the additional D. Bernoulli equation for the total hydrodynamic pressure. The graphic was given in dimensionless coordinates, and at the first approximation, it allowed the design organizations to obtain the entire spectrum of flow parameters. However, the adequacy of this solution was rather low and it made the researchers of 2D planar water flows search for new ways to solve the problem. The development of numerical solution of the problem based on integral equation system has not been brought to possibility of practical use in hydraulic construction.

G.A. Lilitsky's method is not universal, it is based on the experimental data processing.

The most promising method of solving problems may be the analytical method using the velocity hodograph plane [4]. However, it has its own drawbacks.

2.4. Description of the analytical methods using the velocity hodograph plane

According to S.A. Chaplygin's method of [10], the system of equations (5) is converted to the variables τ and θ by using a complex differential connection between the physical plane Ψ(XY) and the velocity hodograph plane $\Gamma(\tau,\theta)$:

$$d(x+iy)=\frac{1}{\tau^{1/2}\sqrt{2gH_0}}\left[d\varphi+i\frac{h_0}{H_0(1-\tau)}d\psi\right]\cdot e^{i\theta}, \tag{10}$$

where τ is flow kineticity; θ is angle characterizing the direction of the velocity vector to the longitudinal axis of symmetry of the flow; $i=\sqrt{-1}$ is the imaginary unit. We obtain a system of equations [1, 4]:

$$\begin{cases} \dfrac{\partial\varphi}{\partial\tau}=\dfrac{h_0}{2H_0}\cdot\dfrac{3\tau-1}{\tau(1-\tau)^2}\cdot\dfrac{\partial\psi}{\partial\theta}; \\ \dfrac{\partial\varphi}{\partial\theta}=2\dfrac{h_0}{H_0}\cdot\dfrac{\tau}{1-\tau}\cdot\dfrac{\partial\psi}{\partial\tau}, \end{cases} \tag{11}$$

where $\varphi=\varphi(\tau,\theta)$ is the potential function; $\psi=\psi(\tau,\theta)$ is the current function; the hydro-dynamic pressure is

$$H_0=\frac{V_0^2}{2g}+h_0=\frac{V^2}{2g}+h=H;$$

the flow kineticity parameter is

$$\tau=\frac{V^2}{2gH_0}; \tag{12}$$

the local depth and velocity (average over the live section) is:

$$h=H_0(1-\tau);\;\; V=\tau^{1/2}\sqrt{2gH_0}. \tag{13}$$

The system of equations (11) is already a linear system admitting to obtain analytical solutions. The authors in [4, 9, 18] chose from these solutions the following solution:

$$\psi=A\cdot\frac{\sin\theta}{\tau^{1/2}};\;\; \varphi=A\cdot\frac{h_0}{H_0}\cdot\frac{\cos\theta}{\tau^{1/2}(1-\tau)}, \tag{14}$$

where the constant A is determined according to the theory described in [4, 5]:

$$A=\frac{V_0 b}{2\sin\theta_{\max}};$$

here b is the width of the culvert.

Therefore, by solving the problem in the plane of the velocity hodograph and using (11), we can define the solution over the entire spectrum of flow parameters in the flow plane.

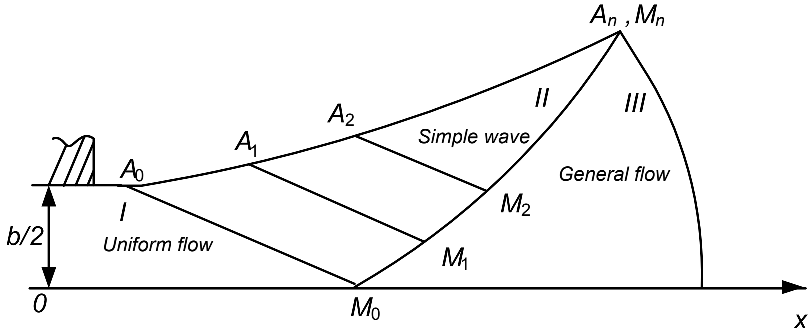
The main necessary requirements to perform boundary conditions of the problem of free flow spreading at $\tau\to 1$ are as follows:

$$h\to 0;\;\; V\to V_{\max}=\sqrt{2gH_0};\;\; h=H_0(1-\tau). \tag{15}$$

There are two appropriate types of flows to fulfill conditions (15): radial spreading of the flow and flow of the type (14).

The additional condition is flow spreading with a free surface curved from the bottom towards the atmosphere along the channel (the axis of symmetry of the flow). This condition follows from experimental studies of flow spreading. The flow of type (14) also corresponds to this condition but the radial spreading of the flow does not. This is why the authors have chosen the scheme (Fig. 2) and the common type flow (14).

However, solution (14) satisfies the boundary conditions on infinity at $\tau \to 1$, but it does not satisfy the boundary conditions at the outlet of the pipe at $\theta = 0,\ \tau_0 = \dfrac{V_0^2}{2gH_0}$.



$A_0M_0,\ A_1M_1$ are characteristics of the 2nd family;

$M_0,\ M_1, \cdots M_n$ are characteristic of the first family; *OX* –flow symmetry axis

Figure 2. Scheme of combining a uniform flow and the flow (14): I – steady-state flow in the pipe; II – simple wave; III – general flow.

The authors tried to eliminate this contradiction by searching for other types of solutions and by switching from a straight edge of the outlet pipe to a curvilinear shape as well. The authors obtained a sufficient adequacy to the real process in terms of the form of the border streamline (but not in all parameters) [5–7]. A detailed study of the theory of 2D planar water flows provided the following conclusion [10, 19]: a uniform flow in a pipe can be combined with a flow of the general type to which the flow belongs (14), only by a simple wave. This is why when using the scheme of free spreading of the flow (region 2, see Fig. 2) it satisfies the boundary conditions at the outlet of the pipe.

2.5. Determination of flow parameters in various areas

Let us consider the region of flow spreading in the plane of the velocity hodograph, see Fig. 3. Here are the following indications:

$M_0\left(\tau_0, 0\right)$ is the point corresponding to the region **I** of a uniform flow current (see Fig. 2) with parameters $\tau_0,\ \theta = 0.$ On the velocity hodograph plane the point M_0 corresponds to the entire region **I** of the uniform flow current.

M_0M_n is the characteristic of the first family, the conjugation line of the flow of the general form (14) **III** and the simple wave flow (see Fig. 2).

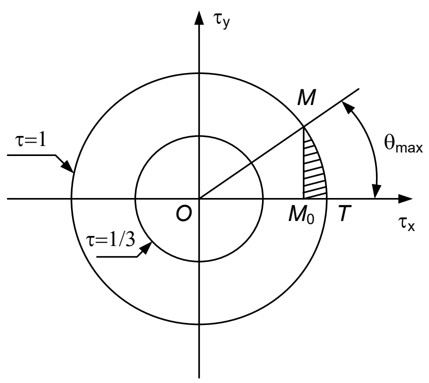


Figure 3. Flow spreading region in the velocity hodograph plane.

1. Determining the boundaries of the uniform flow area.

Point M_0 in the sector I (see Fig. 2) belongs to the characteristic of the 1st family with parameters $\tau_0,\ \theta = 0.$ The wave angle at this point is

$$\alpha_0 = \arcsin\sqrt{\frac{1-\tau_0}{2\tau_0}};\quad A_0M_0 = \frac{b}{2\mathrm{tg}\alpha_0}.$$

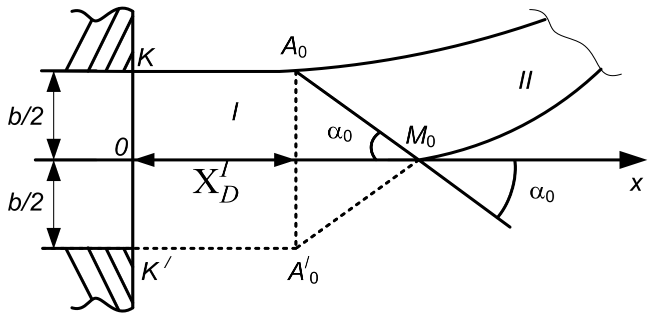


Figure 4. Boundaries of the of uniform flow area.

Since M_0A_0' and M_0A_0 are straight lines, the pentagon is $K'OKA_0M_0A_0'$ in Fig. 4 is symmetrical about the OX axis. Its upper part is a rectangular trapezoid. Thus, with known distance $\mathrm{X}_D^I,$ the geometry and kinematics of sector I of the flow spreading are determined.

To determine the parameter X_D^I we use the formula given in [8]

$$\mathrm{X}_D^I = trunc\left[\frac{\sqrt{F_0-1}}{\sin\theta_{\max}\left(F_0+2\right)}h_0\right]+1, \tag{16}$$

where 1 is the depth measuring unit of h_0. In calculations this is 1 cm, in natural experiments it is 1 m. Formula (16) was derived from the compilation of the structural formula and the determination of the coefficients by the regression analysis method. To derive this formula we used information from 70 experiments, some of which was published in [4].

The flow parameters $V_0,\ h_0,\ \tau_0$ in the closed region I are constant and are defined by formulas (12) and (13).

2. Determining flow parameters in terms of the M_0M_n characteristic. The angle of inclination of the first family characteristicsin the plane of the velocity hodograph has the following form [4, 18]:

$$\theta = \sqrt{3}\cdot\mathrm{arctg}\sqrt{\frac{3\tau-1}{3\left(1-\tau\right)}} - \mathrm{arctg}\left(\sqrt{\frac{3\tau-1}{1-\tau}}\right) + C', \tag{17}$$

where the constant C' is found from the condition of the characteristic's passing through the point M_0:

$$C\,' = \mathrm{arctg}\sqrt{\frac{3\tau_0-1}{1-\tau_0}} - \sqrt{3}\cdot\mathrm{arctg}\left(\frac{1}{\sqrt{3}}\sqrt{\frac{3\tau_0-1}{1-\tau_0}}\right).$$

The angle θ takes the maximum value at the boundary condition

$$\tau\to 1;\quad h = H_0\left(1-\tau\right)\to 0;\quad V = \tau^{1/2}\sqrt{2gH_0}\to\sqrt{2gH_0};\quad \theta\to\theta_{\max}$$

and is equal to

$$\theta_{\max} = C' + \left(\sqrt{3}-1\right)\cdot\frac{\pi}{2}.$$

Note that at $\tau \to 1$ according to the scheme (Fig. 2), the boundary streamline (free boundary) tends to the characteristic $M_0 M_n$ and the points A_n and M_n approach each other, since the wave angle tends to zero:

$$\alpha_n = \arcsin \sqrt{\frac{1-\tau_n}{2\tau_n}} \to 0.$$

The parameters $\tau, \ \theta$ on the characteristic of the 1st family are also parameters of the “simple wave” flow. In a simple wave the characteristics of the second family are straight lines [1].

Given the parameter $\tau_M \in \left[\tau_0, 1\right]$, we define the corresponding angle θ_M by formula (17) and calculate the depths and velocities by formulas (13).

2. Determining the coordinates of the points M_i along the characteristic of the first family, see Fig. 5.

The specific volumetric flow rate coefficient of an arbitrary streamline is equal to:

$$Q = K \frac{b}{2}, \quad 0 \le K \le 1,$$

where K is the flow rate coefficient.

To determine the kineticity τ_M along the characteristic of the 1st family we substitute function (17) into the streamline equation (14) and solve it [18]:

$$\frac{\sin \theta_M}{\tau_M^{1/2}} = K \sin \theta_{\max}. \tag{18}$$

We find τ_M and then θ_M.

To find the coordinates of points M_i, we use the relation equation (10) between the physical plane Ψ(XY) and the plane of the velocity hodograph $\Gamma\left(\tau, \theta\right)$. Separating the imaginary and real parts in (10) and considering that $d\psi = 0$ along the streamline, we obtain

$$\begin{cases} dx = \dfrac{d\varphi}{\tau^{1/2}\sqrt{2gH_0}} \cdot \cos\theta; \\ dy = \dfrac{d\psi}{\tau^{1/2}\sqrt{2gH_0}} \cdot \sin\theta. \end{cases} \tag{19}$$

From the equation (18) we have

$$\sin \theta_M = K \cdot \tau_M^{1/2} \sin \theta_{\max}. \tag{20}$$

Then, by differentiating both parts of the expression (20) and omitting the index "M", we obtain:

$$\cos\theta d\theta = \frac{1}{2} K \cdot \frac{d\tau}{\sqrt{\tau}} \sin \theta_{\max} \tag{21}$$

or

$$d\theta = \frac{1}{2} K \cdot \frac{d\tau}{\sqrt{\tau}} \frac{\sin \theta_{\max}}{\cos \theta}. \tag{22}$$

To determine $d\varphi$ we use the expression for the potential function in equations (14)

$$\varphi = A \cdot \frac{h_0}{H_0} \cdot \frac{\cos\theta}{\tau^{1/2}\left(1-\tau\right)}. \tag{23}$$

Differentiating $\varphi = \varphi(\tau, \theta)$, we obtain

$$d\varphi = \frac{Ah_0}{H_0}\left[\frac{-\sin\theta d\theta}{\tau^{1/2}(1-\tau)} + \frac{\cos\theta(3\tau-1)d\tau}{2\tau^{3/2}(1-\tau)^2}\right]. \tag{24}$$

Considering expressions (20)–(24) we can transform the system (19) to:

$$\begin{cases} dx = f_1(\tau)d\tau; \\ dy = f_2(\tau)d\tau, \end{cases} \tag{25}$$

where $f_1(\tau),\ f_2(\tau)$ are the functions of the flow kinetics parameter defined after transformations.

Integrating the system (25) at the initial values we obtain the values of the coordinates of the selected flow point:

$$X = X_{M_0} + \int_{\tau_0}^{\tau} f_1(\tau)d\tau; \quad Y = Y_{M_0} + \int_{\tau_0}^{\tau} f_2(\tau)d\tau.$$

Thus, we find the coordinates of the point M that is located of the intersection of the streamline with the flow rate coefficient K and the characteristics of the first family in the planar flow.

3. Determining flow parameters in the flow region III (see Fig. 2).

We have to solve the system of equations in the plane of the velocity hodograph to determine the parameters θ, τ at the point of intersection of an arbitrary streamline and an arbitrary equipotential:

$$\begin{cases} A\dfrac{\sin\theta}{\tau^{1/2}} = K\sin\theta_{\max}; \\ \dfrac{\cos\theta}{\tau^{1/2}(1-\tau)} = \dfrac{\cos\theta_x}{\tau_M^{1/2}(1-\tau_M)}, \end{cases} \tag{26}$$

where K is given; θ_M, τ_M are parameters at the point of intersection of the characteristics of the first family with the streamline determined by the flow rate coefficient K. The solution of the system (26) is reduced to the solution of the cubic equation [4].

4. Determining the parameters on the rectilinear characteristics of the second family, in a simple wave II and determining the coordinates of the points $A_1,\ A_2, \ldots,\ A_n$ (Fig. 2).

Parameters τ_M, θ_M change along the characteristic of the 1st family, but not in simple waves – section II (from the characteristic of the 1st family up to the flow boundary). In other words, the boundaries of the flow and characteristics of the 1st family have the same look in the velocity hodograph plane [1, 4].

Since the characteristics of the 2nd family are straight lines with constant parameters τ_M, θ_M, they will have the same values of the parameters $\tau_A = \tau_M$, $\theta_A = \theta_M$ in the points $A_0,\ A_1, \ldots,\ A_n$ of their intersection with the flow boundary. As soon as the value of the kinetic parameter τ_A is determined it is possible to find the velocity and depth of the flow at the point A_i using the formulas (13).

To construct the boundary streamline, we have parameters $\tau_A,\ \theta_A,$ and the direction of the segments $A_i A_{i+1}$ should be taken along the corresponding streamline passing through these points on the characteristic, with the wave angle taken into account [4].

$$\alpha_i = \arcsin\left(\sqrt{\frac{1-\tau_i}{2\tau_i}}\right). \tag{27}$$

To determine the coordinates of the points of the boundary streamline, we write a system of equations in which the first equation determines a pencil of lines passing through the point M_i, and the second equation is the equation of a circle centered at the point M_i.

The coordinates of the boundary streamline point are the solution of this system of equations, i.e. intersection point of a pencil of lines and a circle.

Omitting the index "$i+1$", we have

$$\begin{cases} (Y_A - Y_M) = \operatorname{tg}(\theta - \alpha)(X_A - X_M); \\ (X_A - X_M)^2 + (Y_A - Y_M)^2 = \rho_{AM}^2. \end{cases} \quad (28)$$

Obviously, the following conditions must be met:

$$Y_A > Y_M > \frac{b}{2}; \ \ Y_{A_{i+1}} > Y_{A_i}; \ \ X_A > X_{A_0}; \ \ X_{A_{i+1}} > X_{A_i}.$$

To determine the coordinates of points lying on the boundary streamline we solve the system of equations (28). As a result, we obtain:

$$\begin{cases} Y_A = Y_M + k\left|X_A - X_M\right|; \\ X_A = X_M - \dfrac{\rho_{AM}}{\sqrt{1+k^2}}, \end{cases} \quad (29)$$

where $k = \left|\operatorname{tg}(\theta - \alpha)\right|$.

Let us consider a model of a non-stationary potential 2D planar open high-velocity water flow of an ideal fluid with free spreading back of a culvert, as a test example. The validation of the model is performed on the basis of real experimental data published in [4]. Data on spreading coordinates, depth and flow velocity are given in Table 1.

Table 1. Experimental data on the coordinates of the water flow, its depth and velocity.

X_E (cm)	9	24	44	64	71
Y_E (cm)	9	38	59	76	80
V (cm/s)	151.928	186.461	191.243	192.714	194.49

The flow at the outlet of the pipe has the following initial parameters:

- initial flow velocity (at the outlet of a pipe) $V_0 = 147.654$ (cm/s);
- initial flow depth relative to the bottom $h_0 = 9.27$ (cm);
- acceleration of gravity $g = 981$ (cm/s^2);
- pipe width $b = 16$ (cm);
- flow rate at the outlet of a culvert $Q = 2.19 \cdot 10^4$ (cm^3/s);
- relative flow expansion $\beta = 5$.

3. Results and Discussion

3.1. Parameters in the area of uniform flow and on the characteristic of the 1st family

According to the above presented algorithm, we find:

- Froude criterion $F_0 = 2.397$ (see formula (16));
- hydrodynamic head $H_0 = 20.382$ (cm);

- initial flow kinetics $\tau_0 = 0.545$;
- length of the inertial front $X_D^I = 3$ (cm) [8];
- wave angle at the outlet of the pipe is $\alpha_0 = 0.702\ radian$ or $\alpha_0 = 40°23$;
- angle of inclination of the fluid flow velocity vector to the *OX* axis at infinity $\theta_{\max} = 0.981\ radian$ or $\theta_{\max} = 56°23$;
- distance along the flow symmetry axis from the end of the inertial section to the $M_0 : AM_0 = 9.457$ (cm);
- straight line length of a segment of the characteristic of the second family between the points A_0 and $M_0 \cdot A_0M_0 = 12.387$ (cm).

We divide the width of the flow pipe $b/2$ into 40 parts with step length $\Delta Y = 0.2.$ We find the flow parameters at the points of intersection of a particular streamline with the characteristic of the first family, see equation (18). Table 2 shows the calculated values of the flow coefficient K, kineticity τ_M, the angle of inclination of the velocity vector $\theta_M\left(radian\right)$, abscissa $X_M\left(\mathrm{cm}\right)$ and ordinate $Y_M\left(\mathrm{cm}\right)$ at the points on the characteristic of the 1st family.

Table 2. Values of the flow parameters on the characteristic of the 1st family.

No. step	K_i	τ_{Mi}	θ_{Mi}	X_{Mi}	Y_{Mi}
0	0	0.545	0	0	0
1	0.25	0.559	0.016	12.633	$4.162\cdot10^{-4}$
2	0.05	0.574	0.031	12.899	$1.77\cdot10^{-3}$
3	0.075	0.588	0.048	13.185	$4.234*10^{-3}$
⋮	⋮	⋮	⋮	⋮	⋮
28	0.7	0.926	0.594	39.34	6.273
29	0.725	0.936	0.623	43.235	8.757
⋮	⋮	⋮	⋮	⋮	⋮
33	0.825	0.972	0.742	73.489	47.873
34	0.85	0.979	0.774	89.655	85.193
⋮	⋮	⋮	⋮	⋮	⋮

Next, we constructed a grid in the general flow region and the parameters are determined according to the equations (26). We do not present them here.

3.2. Determining the coordinates of points along the boundary streamline A_i

Since the parameters τ_M, θ_M do not change in simple waves, we assume that $A = \tau_M$, $\theta_A = \theta_M$.

We calculate the wave angle $\alpha\left(radian\right)$ (see expression (27)) for segments of the straight lines A_iM_i. From the system of equations (29) we find the radii of the circles $\rho\left(\mathrm{cm}\right)$ with centers at the points M_i and define the coordinates of the points A_i on the boundary streamline.

The adequacy of the model was evaluated in terms of the error of the abscissa, ordinate, and velocity compared to the experimental data. The calculation results are summarized in Table 3.

Table 3. Values of the flow parameters along the boundary streamline.

No. step	α_i	ρ_i	X_{Ai}	X_E	δX, %	Y_{Ai}	Y_E	δY, %	V_{Ai}	V_E	δV, %
0	0.702	12.387	0	0		8	8	0	147.654		
1	0.678	12.989	9.545	9		9.457	9	0.455	149.561	151.928	1.558
2	0.656	13.303	10.643			10.324			151.45		
3	0.634	13.64	11.75			11.458			153.321		
⋮	⋮	⋮			⋮	⋮	⋮				
28	0.201	61.37	23.337	24	2.761	35.771	38	6.233	192.435	186.461	3.204
29	0.185	70.962	25.767		5.671	58.644		0.607	193.504	191.243	3.067
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮				
33	0.121	158.398	46.495	44	8.654	73.13	59	3.924	197.109	192.714	2.649
34	0.105	212.027	58.462	64	9.855	89.332	76	10.446	197.82	194.49	2.034
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

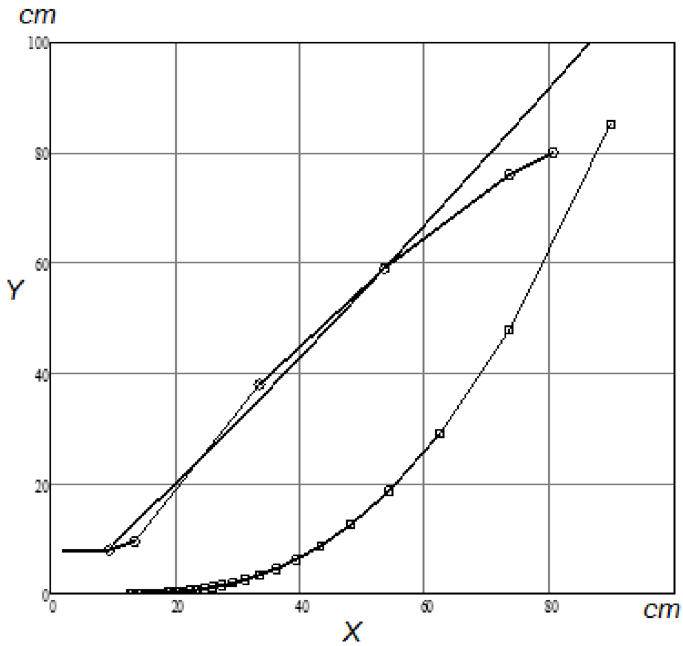


Figure 5. The graphs of the boundary streamlines obtained by the presented method (solid line) and experimental data (line labeled with circles), as well as the characteristic curves of the 1st family (line labeled with squares).

Fig. 5 shows the curves of the extreme streamline obtained by the presented algorithm (solid line) and experimental data (labeled with circles), as well as the characteristic curve of the 1st family (labeled with square markers).

Detailed development of programs was published by the authors in [22, 23].

This work develops the methods for the analytical study of problems of technical fluid and gas mechanics [1, 18–21], using a technique for solving nonlinear problems similar to those proposed in [24–27].

4. Conclusions

1. The article formulates a mathematical model of a 2D high-velocity planar flow with justification and consideration of some physical assumptions. The boundary problem of free spreading of the flow in terms of the flow has been set. A system of equations for the motion of a water flow in a dimensionless form has been obtained in two ways. The criteria influencing the process of flow spreading were revealed, their versatility was proved. The conclusion is drawn that the known methods based on Sherenkov's universal graph do not provide sufficient adequacy for the solution to the free spreading water flow problem.

2. The theory in the article agrees with the main points of the theory of conjugation of 2D potential flows. The proposed method for solving the problem includes both previously known and new points. The novelty is the division of the flow into three main sections: a uniform flow, a general flow and a conjugating flow of a simple wave. The use of simple waves to determine the coordinates of the extreme line of the water flow is also new.

3. Using Chaplygin's method made it possible to obtain an analytical method for solving the problem of flow free spreading in the plane of the velocity hodograph. The method is effective, unambiguous and has no singularities and discontinuities at the outlet of a pipe.

4. A test case was calculated. The parameters of the flow and its coordinates on the characteristic of the 1st family and the boundary streamline at the points coinciding with the natural experiment are given (Table 2 and Table 3). Relative errors are calculated. A more complete calculation is presented in the form of a graph in Fig. 5. The adequacy of the obtained flow model to the natural experiment was proved. The calculation error is less than 10 %, which is quite acceptable.

References

1. Yemtsev, B.T. Dvuhmernye burnye potoki [2D High-Velocities Flows]. Moscow, 1967. 212 p. (rus)
2. Sherenkov, I.A. O planovoj zadache rastekaniya strui burnogo potoka neszhimaemoj zhidkosti [On the Planned Problem of the Spreading of a Jet of a Rough Flow of an Incompressible Liquid]. Izv. Academy of Sciences of the USSR. REL. 1958. 1. Pp. 72–78. (rus)
3. Esin A. I. Hydrodynamics equation of two-dimensional open flows in natural coordinates. Fluid Mechanics. Soviet Research (Script a Technical, USA). 1984. 13(6).
4. Kokhanenko, V.N., Volosukhin, Ya.V., Lemeshko, M.A., Papchenko, N.G. Modelirovanie burnyh dvuhmernyh v plane vodnyh potokov [Modeling of Stormy Two-Dimensional in Terms of Water Flows]. Rostov-on-Don: YuFU Publ., 2013. 180 p. (rus)
5. Duvanskaya, E.V. Obshchiy metod analiticheskogo resheniya zadach planovoy gidravliki spokoynykh potokov [Generalized method of analytical solution of precritical flows in problems of planned hydraulics]. Izv. VUZov. The North-Caucasus Region. Technical Sciences. 2011. 3. Pp. 91-93. (rus)
6. Kosichenko, N.V. Analiz izucheniya i utochneniya metodov svobodnogo rastekaniya potoka za beznapornymi vodopropusknymi otverstiyami [The analysis of studying and specification of methods of free spreading of a stream behind free-flow water throughput openings]. Vestnik SGAU. 2011. 9. Pp. 27–33. (rus)
7. Papchenko, N.G The general technology of solving practical problems of hydraulics of two-dimensional in terms of stationary turbulent water flows analytical method using the hodograph plane speed. Proceedings of Voronezh State University. Series: Physics. Mathematics. 2014. 2. Pp. 162–166. (rus)
8. Kelekhsaev, D.B. Calculation of the rapid flow of water at the outlet of the round pipe in the downstream of the culverts. Construction and Architecture. 2018. 3 (20). Pp. 29–34. DOI: 10.29039/article_5bee8ada6e3492.04272155
9. Burtseva, O.A., Alexandrova, M.S. Water flow parameters on the symmetry axis and extreme line of current. Vestnik MGSU [Monthly Journal on Construction and Architecture]. 2023. 18(8). Pp. 1262–1271. DOI: 10.22227/1997-0935.2023.8.1262-1271
10. Alexandrova, M.S. Method of analogies between hydraulics of two-dimensional water flows and gas dynamics. Construction and Architecture. 2020. 2(27). Pp. 49–52. DOI: 10.29039/2308-0191-2020-8-2-49-52
11. Nikora, V.I., Stoesser, T., Cameron, S.M., Stewart, M., Papadopoulos, K., Ouro, P., McSherry, R., Zampiron, A., Marusic, I. Falconer, R.A. Friction factor decomposition for rough-wall flows: theoretical background and application to open-channel flows. Journal of Fluid Mechanics. 2019. 872. Pp. 626–664. DOI: 10.1017/jfm.2019.344
12. Aranda, J., Beneyto, C., Sánchez-Juny, M., Bladé, E. Efficient design of road drainage systems. Water. 2021. 13. DOI: 10.3390/w13121661
13. Sanz-Ramos, M., Bladé, E., González-Escalona, F., Olivares, G., Aragón-Hernández, J.L. Interpreting the manning roughness coefficient in overland flow simulations with coupled hydrological-hydraulic distributed models. Water. 2021. 13. 3433. DOI: 10.3390/w13233433
14. Anees, M.T., Abdullah, K., Nordin, M.N.M., Rahman, N.N.N.A., Syakir, M.I., Kadir, Mohd.O.A. One- and two-dimensional hydrological modelling and their uncertainties. Flood Risk Management. InTech. 2017. DOI: 10.5772/intechopen.68924
15. Nematollahi, B., Abedini, M.J. Analytical solution of gradually varied flow equation in non-prismatic channels. Iranian Journal of Science and Technology – Transactions of Civil Engineering. 2020. 44. Pp. 251–258. DOI: 10.1007/s40996-019-00316-5
16. Castro-Orgaz, O., Cantero-Chinchilla, F.N. Non-linear shallow water flow modelling over topography with depth-averaged potential equations. Environmental Fluid Mechanics. 2020. 20. Pp. 261–291. DOI: 10.1007/s10652-019-09691-z
17. Li, J., Li, S.S. Near-bed velocity and shear stress of open-channel flow over surface roughness. Environmental Fluid Mechanics 2020. 20. Pp. 293–320. DOI: 10.1007/s10652-019-09728-3
18. Evtushenko, S.I. A nonlinear system of differential equations in supercritical flow spread problem and its solution technique. Axioms 2023. 12(1). 11. DOI: 10.3390/axioms12010011
19. Alexandrova, M.S. Simple waves in the theory of two-dimensional water flows and the scheme of their use for free flow spreading. Construction and Architecture. 2020. 3(28). Pp. 47–50. DOI: 10.29039/2308-0191-2020-8-3-47-50
20. Orlov, V., Gasanov, M. Analytic approximate solution in the neighborhood of a moving singular point of a class of nonlinear equations. Axioms. 2022. 11(11). 637. DOI: 10.3390/axioms11110637
21. Orlov, V., Gasanov, M. Technology for obtaining the approximate value of moving singular points for a class of nonlinear differential equations in a complex domain. Mathematics. 2022. 10 (21). 3984. DOI: 10.3390/math10213984
22. Papchenko, N.G. Certificate of state registration of computer programs No. 2014611308. 2014.
23. Aleksandrova, M.S. Determination of flow parameters along the border current line. Certificate of state registration of computer programs. No 2022666655. 2022.

24. Orlov, V., Chichurin, A. About analytical approximate solutions of the van der pol equation in the complex domain. Fractal and Fractional 2023. 7(3). 228. DOI: 10.3390/fractalfract7 030228
25. Orlov, V. Moving singular points and the van der pol equation, as well as the uniqueness of its solution. Mathematics. 2023. 11(4). 873. DOI: 10.3390/math11040873
26. Orlov, V., Chichurin, A. The influence of the perturbation of the initial data on the analytic approximate solution of the van der pol equation in the complex domain. Symmetry. 2023. 15(6). 1200. DOI: 10.3390/sym15061200
27. Orlov, V. Dependence of the Analytical Approximate Solution to the Van der Pol Equation on the Perturbation of a Moving Singular Point in the Complex Domain. Axioms. 2023. 12(5). 465. DOI: 10.3390/axioms12050465

Information about authors:

Olga Burtseva, *PhD in Technical Sciences*
ORCID: https://orcid.org/0000-0003-4288-8709
E-mail: kuzinaolga@yandex.ru

Sergej Evtushenko, *Doctor of Technical Sciences*
ORCID: https://orcid.org/0000-0003-3708-380X
E-mail: evtushenkosi@mgsu.ru

Viktor Kokhanenko, *Doctor of Technical Sciences*
ORCID: https://orcid.org/0000-0002-9962-3966
E-mail: kokhanenkovn@mail.ru

Received 18.05.2021. Approved after reviewing 18.12.2023. Accepted 10.01.2024.