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Providing free vibrations and stability of a multi-span beam under temperature changes by selecting the support system

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Abstract. This study presents a newly developed method for reasonable selection of boundary conditions and number of pinned intermediate supports for a straight multi-span beam. This method might help to obtain the required values of the first frequency of free vibrations and the critical load from the action of axial force, resulting from the changing the beam temperature. The method is based on known concepts of beam vibration and stability theories and uses support coefficients as a criterion for selecting the appropriate support system for a multi-span beam. These coefficients are obtained by solving the corresponding differential equations of the dynamic behavior of the beam and are determined only by the support conditions. Comparative calculations of the straight pipeline using the developed and finite element methods for beam and shell models were carried out, which showed good convergence. Normalization of the values of the support coefficients allowed to combine both conditions, for the first natural vibration frequency and the first critical force, and express it as a single criterion for the selection of the support system. The selection of the support system is shown as three general methods of fixing multi-beam beams with a constant span length. This approach can be applied to any straight beams and support conditions for which support coefficient values are known. To this end, a general algorithm for selecting a support system with known support coefficients and requirements for their normalization is given. The results obtained can be used in the calculation and design of any multi-span beam structures to control the values of their free vibration frequencies and stability by selecting an appropriate support system during the engineering design process.

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1. *Introduction*

A large number of extended periodic structures are subject to forced vibrations and temperature changes. These include pipelines, oil lines, steam lines, railway rails, pull rods, cables, beams, etc. Serviceability conditions for such structures ensure the permissible values of the first natural frequency of vibration f_1 and the first critical force P_{cr1} or temperature ΔT_{cr1} at which the loss of stability occurs:

$$f_1 \geq [f],\ P_{cr1} \geq [P] \text{ or } \Delta T_{cr1} \geq [\Delta T]. \quad (1)$$

In this paper, these multi-span structures are modeled on the basis of Euler–Bernoulli beam theory for cases of transverse free vibrations and stability loss. Theoretical foundation for calculating the vibrations and stability of multi-span beams have been described in many papers. This beam dynamic behavior is commonly calculated by partial differential equations with specified boundary conditions, which are determined by the supports of the beam [1–39]. As it is difficult to obtain an analytical solution to this problem, numerical calculation methods and specialized computer programs are used. They are mostly based on the finite element method [40–42]. At the same time, numerical methods allow to obtain only individual particular solutions. This complicates their use for quick qualitative assessment of structural decisions in order to meet serviceability conditions (1).

Recently, applied calculation methods and various reference books on dynamics were developed to help engineers conduct the necessary calculations without setting up and solving differential equations [43, 44]. However, almost all the literature on vibrations and stability is focused only on assessing the dynamic behavior of already existing structures with given supports without the possibility of their design calculation. Nevertheless, engineers mostly need to determine the required support system with boundary conditions and number of pinned intermediate supports for a given beam to ensure the serviceability conditions (1).

The literature on vibration protection [45–51] mainly considers the following vibration control methods: vibration isolation, additional damping, balancing, etc. In the optimal design of beam structures and topology optimization [52–76], the authors focus on making changes to the beam itself, wherein changing variables must be continuous and smooth, which is difficult for discrete changes in the number and type of supports. It is also true for beam stability calculations.

This paper is aimed at a new approach for a reasonable selection of a support system for a straight multi-span beam that provides the required values of the first natural vibration frequency and first critical force taking into account the temperature.

To achieve this goal, the following tasks were solved;

1) to develop a design calculation approach for a reasonable selection of a support system for a straight multi-span beam that provides its required dynamic behavior (the first natural vibration frequency and the first critical force) taking into account the temperature;
2) to get a single criterion based on support coefficients for the design approach;
3) to propose a method, which makes it possible for an engineer to quickly and reasonably perform a multifactor assessment or design of multi-span beams to achieve the required dynamic behavior;
4) to justify the method by comparative calculations of a pipeline using different approaches.

2. Methods

The equation of free vibrations of a beam takes into account the action of the axial force P (Fig. 1a) and thus looks as follows [2, 3]:

$$EJ_{\min}\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} = 0. \quad (2)$$

As a function of the deflection $y(x,t)$ for the bending mode of free vibrations at the first natural frequency, the following equation should be considered:

$$y_1(x,t) = A\sin\left(\frac{x\pi}{l}\right)\sin(\omega t). \quad (3)$$

For an unambiguous solution of equation (2), setting four boundary conditions that reflect the beam restraint conditions on supports is necessary.

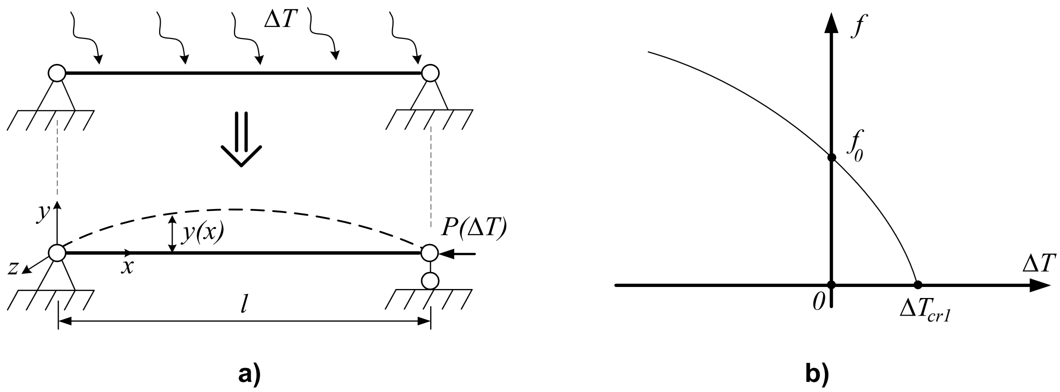

Figure 1. Effect of the beam temperature on its dynamic state: a) temperature design diagram; b) frequency-temperature curve.

By substituting the deflection function (3) into equation (2) and temporarily taking $P = 0$ as well as the hinge restraint conditions, the solution for the first natural vibration frequency is calculated as follows:

$$f_1 = \frac{\alpha^2}{2\pi l^2} \cdot \sqrt{\frac{EJ_{\min}}{m}}. \qquad (4)$$

When the axial compressive force P acts on the beam, the first natural vibration frequency can be defined by Galef's formula [78–80]:

$$f_{1(P<>0)} = f_{1(P=0)} \cdot \sqrt{1 - \frac{P}{P_{cr1}}}. \qquad (5)$$

The axial force P is calculated through the temperature by equation [37]:

$$P = \alpha_t \cdot \Delta T \cdot ES. \qquad (6)$$

The buckling force P_{cr1} for the first instability mode is calculated as follows [37]:

$$P_{cr1} = \frac{\pi^2 EJ_{\min}}{\mu^2 \cdot l^2}. \qquad (7)$$

A typical curve of the dependence of the first natural vibration frequency on temperature is shown in Fig. 1b. By combining equations (4–7), the condition for the first natural vibration frequency of the beam is calculated, taking into account its temperature and the support restraints:

$$f_1(\Delta T) = \left(\frac{\alpha}{\pi l}\right)^2 \cdot \sqrt{\frac{E}{4m}\left(\pi^2 J_{\min} - \mu^2 \cdot l^2 \cdot \alpha_t \cdot \Delta T \cdot S\right)} \geq [f]. \qquad (8)$$

The beam buckling corresponds to the case of its zero natural vibration frequency (Fig. 1b). In equation (8), this corresponds to the equality to zero of the expression in brackets under the root, which makes it possible to use this dependence for assessing stability. For example, the first critical temperature of a beam is defined as follows:

$$\Delta T_{cr1} = \frac{\pi^2 J_{\min}}{\mu^2 \cdot l^2 \cdot \alpha_t \cdot S} \geq [\Delta T]. \qquad (9)$$

The obtained analytical dependence (8) ensures the fulfillment of all serviceability conditions (1).

2.1. Structural Design Approach

Let us consider the values of the support coefficients α and μ (Table 1) for three common support systems (Fig. 2) for multi-span beams [43, 44].

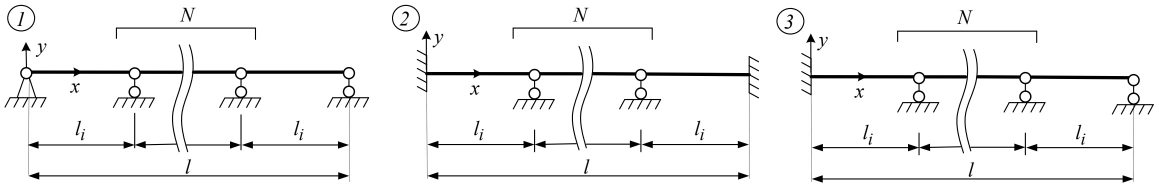

Figure 2. Patterns of beam support systems.

Table 1. Original values of the support coefficients α and μ.

Pattern no.	Coefficient	Number of intermediate supports, *N*										
		0	1	2	3	4	5	6	7	8	9	10
1	α						3.1416					
	μ						1					
2	α	4.730	3.927	3.557	3.393	3.310	3.260	3.230	3.210	3.196	3.186	3.180
	μ	0.5	0.699	0.814	0.879	0.917	0.939	0.954	0.964	0.971	0.977	0.978
3	α	3.927	3.393	3.261	3.210	3.186	3.173	3.164	3.159	3.156	3.153	3.151
	μ	0.7	0.879	0.939	0.964	0.977	0.983	0.988	0.99	0.992	0.994	0.996

If all intermediate supports (Fig. 2) are located equidistant from each other, the length of each span is calculated as follows:

$$l_i = \frac{l}{N+1}. \quad (10)$$

The values of the support coefficients α and μ in Table 1 make it possible to control the dynamic behavior of the beam by selecting their required values.

2.2. Normalization of Support Coefficients

The serviceability condition (8) includes two support coefficients α and μ, but it is necessary to derive a single criterion for the selection method of the appropriate support system. For this purpose, we normalize the initial values of the support coefficients in Table 1 for all patterns (Fig. 2). To do this, it is necessary to divide the initial values of the coefficients (Table 1) by the support coefficients of free beam and take into account equation (10):

$$\alpha' = \left(\frac{\alpha \cdot (N+1)}{\pi}\right)^2, \ \mu' = \left(\frac{N+1}{\mu}\right)^2. \quad (11)$$

Normalized values of the support coefficients α' and μ' for the patterns in Fig. 2 are shown in Table 2.

Table 2. Normalized values of the support coefficients α' and μ'.

Pattern no.	Coefficient	Number of intermediate supports, N										
		0	1	2	3	4	5	6	7	8	9	10
1	α'	1	4	9	16	25	36	49	64	81	100	121
	μ'	1	4	9	16	25	36	49	64	81	100	121
2	α'	2.267	6.250	11.54	18.66	27.75	38.77	51.80	66.82	83.83	102.8	124.0
	μ'	4	8.187	13.58	20.71	29.73	40.83	53.84	68.87	85.91	104.8	126.5
3	α'	1.563	4,666	9.697	16.71	25.72	36.72	49.70	64.71	81.75	100.7	121.7
	μ'	2.041	5.177	10.21	17.22	26.19	37.26	50.20	65.30	82.31	101.2	122.0

Now, the new support coefficients α' and μ' can be compared as both are in the numerator and already take into account the number of intermediate supports.

2.3. Single Criterion for the Structural Design Approach

The analysis of the data from Table 2 shows that the new values of the support coefficients are close to each other for each support pattern, and at $N > 2$ the difference in their values is within 2–3 %.

Moreover, with the increase in N, this difference rapidly decreases because the initial support coefficients (Table 1) rapidly converge to the characteristic constants:

$$\alpha = \pi,\ \mu = 1 \text{ at } N \to \infty. \tag{12}$$

This allows us to introduce a single criterion for the selection of the support system:

$$\alpha_{\min} = \alpha' = \mu'. \tag{13}$$

If condition (13) is substituted in equation (8), then we get a square equation relative to the sought coefficient $\alpha_{\min}$:

$$-\alpha_{\min}^2 \cdot \pi^2 EJ_{\min} + \alpha_{\min} \cdot l^2 \alpha_t \Delta TES + 4l^4 m\left[f\right]^2 = 0. \tag{14}$$

The solution of quadratic equation (14) with the required root sign is as follows:

$$\alpha_{\min} = C_{\Delta T} + \sqrt{C_{\Delta T}^2 + \frac{4m\left[f_1\right]^2 l^4}{\pi^2 EJ_{\min}}}, \tag{15}$$

where $C_{\Delta T}$ is the temperature effect coefficient:

$$C_{\Delta T} = \frac{\alpha_t \cdot \Delta T \cdot Sl^2}{2\pi^2 J_{\min}}. \tag{16}$$

If the temperature effect is irrelevant, we assume $\Delta T = 0,$ and the condition (15) is simplified:

$$\alpha_{\min} = \frac{2l^2\left[f_1\right]}{\pi}\sqrt{\frac{m}{EJ_{\min}}}. \tag{17}$$

To ensure all operability conditions (1), it is necessary to select from Table 2 such pattern and number of intermediate supports for which values of support coefficients α' and γ' are equal or more than the calculated value:

$$\min\left(\alpha',\ \mu'\right) \geq \alpha_{\min} = C_{\Delta T} + \sqrt{C_{\Delta T}^2 + \frac{4m\left[f_1\right]^2 l^4}{\pi^2 EJ_{\min}}}. \tag{18}$$

The selected support system for a beam with the corresponding values of the coefficients α' and μ' ensure the simultaneous fulfillment of both serviceability conditions (1), which can be verified by checking the first vibration frequency and the first critical force or temperature using the new dependencies:

$$f_1\left(\Delta T\right) = \frac{\alpha'}{2l^2} \cdot \sqrt{\frac{E}{m}\left(\pi^2 J_{\min} - \frac{1}{\mu'} \cdot l^2 \cdot \alpha_t \cdot \Delta T \cdot S\right)} \geq \left[f\right]; \tag{19}$$

$$P_{cr1} = \frac{\mu' E}{l^2}\left(\pi^2 J_{\min} - \frac{4l^4\left[f\right]^2 m}{\alpha'^2 E}\right) \geq \left[P\right]; \tag{20}$$

$$\Delta T_{cr1} = \frac{\mu'}{l^2 \cdot \alpha_t \cdot S}\left(\pi^2 J_{\min} - \frac{4l^4\left[f\right]^2 m}{\alpha'^2 E}\right) \geq \left[\Delta T\right]. \tag{21}$$

If we take the condition $\left[f\right] = 0$ in (19–21), the form of these equations is almost identical to their original version (4, 7, 9).

2.4. Algorithm for a Reasonable Selection of Support System for a Beam

The proposed design approach for normalizing the values of the support coefficients allows implementing the method for a reasonable selection of the support system for a beam to ensure the serviceability conditions (1). This method consists of the following steps:

1. Preparation of a set of possible patterns of beam support systems (as in Fig. 2).
2. Determination of the support coefficients α and μ for each pattern from the selected set based on the reference literature or by calculation.
3. Normalization of the values of the support coefficients α and μ for each pattern so that they take close values: $\alpha' \approx \mu'$.
4. Calculation of the minimum required value of the support coefficient $\alpha_{\min}$ (15).
5. Selection of a pattern from the obtained set so that the following condition is satisfied:

$$\min\left(\alpha',\ \mu'\right) \geq \alpha_{\min}.$$

Let us consider an example of calculating an extended structure by the developed method.

3. Results and Discussions

Consider a straight pipeline with a circular cross-section and the following characteristics: length l = 1.5 m, outer diameter D = 15 mm, wall thickness t = 1 mm, material: aluminum alloy E = 7.1 × 10^5 MPa, density ρ = 2,770 kg/m^3, and CTE α_t = 2.3 × 10^{-5}. Initially, the beam is rigidly fixed at both ends without intermediate supports. The objective is to select a support system that will provide the first natural vibration frequency $\left[f_1\right]$ = 250 Hz at a temperature ΔT = 90 °C. We perform the calculation by the developed method and verify the obtained results by the finite element method using the ANSYS software.

3.1. Analytical Solution by the Developed Method

First, let us check the current dynamic parameters of the beam under the initial restraint conditions. According to Table 2, these correspond to the support coefficients α' = 2.267 and μ' = 4. By dependences (19–21) at ΔT = 0 °C and $\left[f_1\right]$ = 0 Hz, we obtain the following initial dynamic characteristics of the beam:

$$f_1 = 39.76\ \text{Hz}, \quad \Delta T_{cr} = 18.79^{\circ}\text{C}. \tag{22}$$

Instability is sure to arise when heating the beam to T_{cr1} = 18.79 °C, which prevents us from using such initial restraint conditions. We cannot use initial restraint conditions at ΔT = 0 °C as, in this case, the first natural vibration frequency is only f_1 = 39.76 Hz. So it becomes necessary to select another support system. For this, according to the proposed method, it is necessary to calculate the required support coefficient according to equation (15):

$$\alpha_{\min} = 26.76. \tag{23}$$

From Table 2, we select a support system with $\min\left(\alpha',\ \mu'\right) > 26.76;$ for example, Pattern 2 with $N = 4,$ for which the support coefficients are equal:

$$\alpha' = 27.75;\ \mu' = 29.73. \tag{24}$$

Let us check the actual values of the first natural vibration frequency and the critical temperature of the beam using the equations (19) and (21):

$$f_2 = 290.16\ \text{Hz};\ \Delta T_{cr2} = 102.78^{\circ}\text{C}, \tag{25}$$

which fulfill the serviceability conditions of this task ($\left[f_1\right]$ = 250 Hz at ΔT = 90 °C).

In Fig. 3, the two curves show the dependences of the first vibration frequency on temperature for both variants (22, 25).

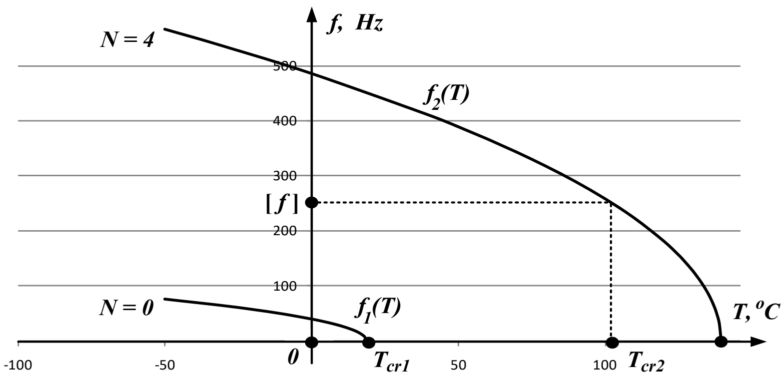

Figure 3. Dependence of the first natural vibration frequency on temperature.

In Fig. 3, the curve $f_2(N)$ for $N = 4$ clearly shows the rise of the first natural vibration frequency of the structure in the entire temperature range under consideration.

3.2. Numerical Solution of the Problem

Let us verify the obtained results using the finite element method (FEM) for the beam and shell models of the straight pipeline. The beam model has 1500 Beam189 finite elements, and the shell model has 14328 Shell281 finite elements. Fig. 4 shows some typical numerical calculation results for the shell model.

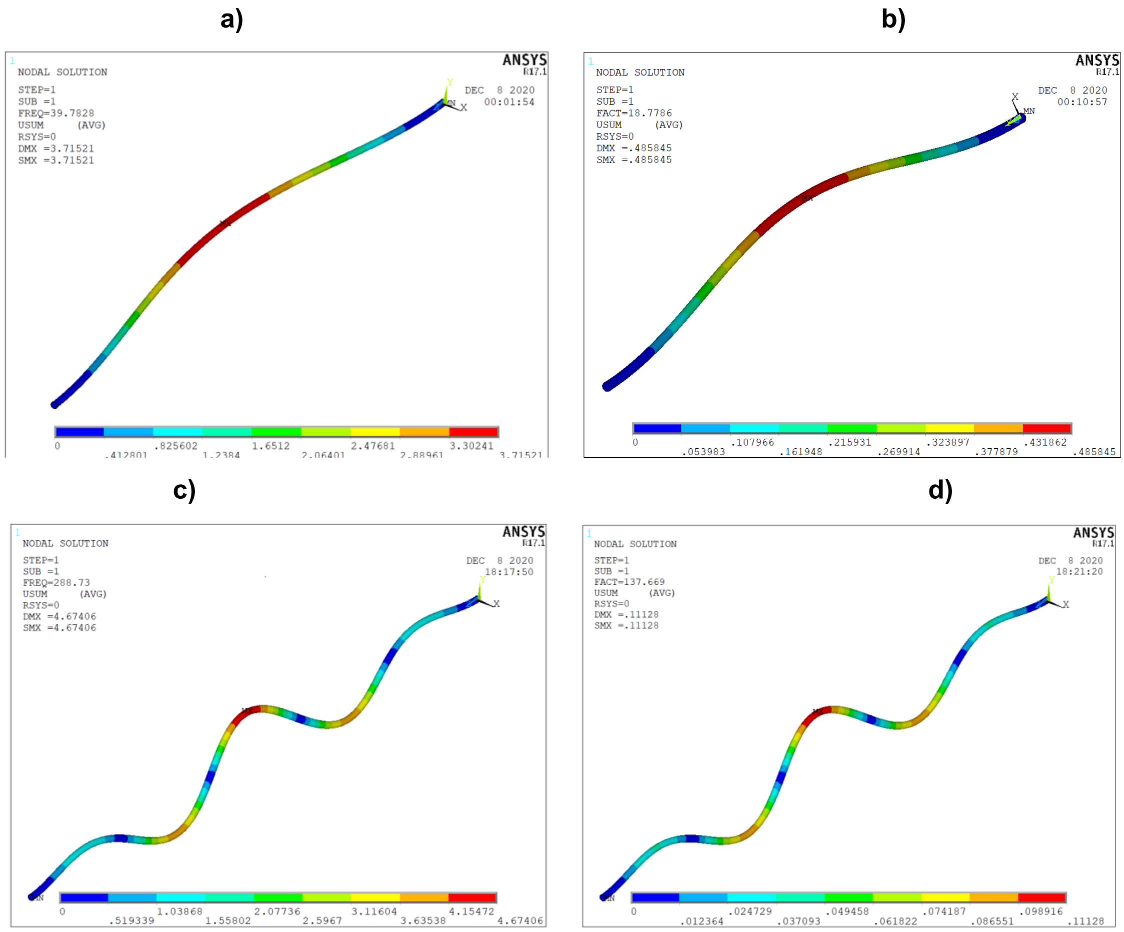

Figure 4. Results of the numerical calculation: a) first vibration mode at N = 0 and ΔT = 0°C; b) instability at T_{cr1}, N = *0* and [*f*] = 0 Hz; c) first vibration mode at N = 4 and ΔT = 90°C*;* d) instability at T_{cr2}, N = 4 and [*f*] = 0 Hz.

The numerical calculation results are summarized in Table 3.

Table 3. Comparison of the calculation results.

	f_1, Hz at ΔT = 0 ˚C	T_{cr1}, ˚C at [f] = 0 Hz	f_1, Hz at ΔT = 90˚C	T_{cr2}, ˚C at [f] = 0 Hz	T_{cr2}, ˚C at [f] = 250 Hz
	N = 0		N = 4		
Developed method	39.76	18.79	290.16	139.62	102.78
FEM, Beam4	39.75	18.78	290.39	139.93	102.91
Deviation, %	0.00961	0.00786	0.0812	0.219	0.119
FEM, Shell281	39.78	18.77	288.73	137.67	100.23
Deviation, %	0.0532	0.0353	0.494	1.40	1.52

The comparison of the calculation results obtained by the proposed method with numerical solutions by FEM shows that the maximum difference is 1.52 %.

Comparison of the proposed method with the papers of other authors shows a complete coincidence for cases when the solution is based on the differential equation (2) and the formula (5), for example, in [78–83]. This approach is correct for the Euler–Bernoulli beam theory, which has some limitations when used. For example, there are some conditions of the beam size ratios [84, 85]. Formula (5) also has its limitations, mostly depending on the ratio [86]:

$$\frac{Pl^2}{EJ_{\min}}. \tag{26}$$

In later works, other authors have developed refined solutions based on more complex dependencies [86–90]. However, the difference in results is less than 5 %, which is acceptable for beam theory, which usually implies a preliminary design calculation.

4. Conclusions

1. The paper developed a design calculation approach for a reasonable selection of a support system for a straight multi-span beam that provides its required dynamic behavior. A new method was developed to help engineers reasonably select the support system for straight multi-span beams, which ensures their serviceability under given requirements for the minimum first natural vibration frequency and the first critical force or temperature.
2. The proposed method has a simple analytical formulation, which also makes it possible for an engineer to quickly and reasonably perform a multifactor assessment or design of any multi-span beam to achieve the required dynamic behavior.
3. The method was justified by the comparative calculations of a pipeline by using the beam and shell models, which showed good convergence for all monitored parameters. Comparison of the method with the works of other authors also showed good convergence in all controlled parameters.
4. The proposed method is to be developed further for the plane multi-support beams consisting of straight and curved sections in order to ensure their dynamic, stressed, and deformed state.

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